

# Generalized Error Exponents For Small Sample Universal Hypothesis Testing

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## Abstract

The small sample universal hypothesis testing problem, where the number of samples  $n$  is smaller than the number of possible outcomes  $m$ , is investigated in this paper. The goal of this work is to find an appropriate criterion to analyze statistical tests in this setting. A suitable model for analysis is the high-dimensional model in which both  $n$  and  $m$  increase to infinity, and  $n = o(m)$ . A new performance criterion based on large deviations analysis is proposed and it generalizes the classical error exponent applicable for large sample problems (in which  $m = O(n)$ ). This generalized error exponent criterion provides insights that are not available from asymptotic consistency or central limit theorem analysis. The results are:

- (i) The best achievable probability of error  $P_e$  decays as  $P_e = \exp\{-(n^2/m)J(1 + o(1))\}$  for some  $J > 0$ .
- (ii) A class of tests based on separable statistics, including the coincidence-based test, attains the optimal generalized error exponents.
- (iii) Pearson's chi-square test has a zero generalized error exponent and thus its probability of error is asymptotically larger than the optimal test.

## Index Terms

Hypothesis testing, large deviations, small sample, separable statistic, error exponent, large alphabet.

## I. INTRODUCTION

As an example of the application of the results, consider the following hypothesis testing problem. An i.i.d. sequence  $\mathbf{Y}_1^n = \{Y_1, \dots, Y_n\}$  with  $Y_i \in [0, 1]$  is observed. There are two hypotheses: Under the null hypothesis  $H_0$ , the probability measure induced by  $Y_i$  is denoted by  $P$ . Under the alternative hypothesis  $H_1$ , it is only known that the probability measure  $Q$  induced by  $Y_i$  satisfies  $Q \in \mathcal{Q}$ . For simplicity of exposition, we assume in this section that  $P$  is absolutely continuous with respect to the Lebesgue measure on  $[0, 1]$ , and the density is positive almost everywhere;  $Q$  is absolutely continuous with respect to  $P$ .

The goal is to design a test  $\phi : [0, 1]^n \rightarrow \{0, 1\}$  with small probabilities of false alarm and missed detection:

$$P_F := P_P\{\phi_n(\mathbf{Y}_1^n) = 1\}, P_M := \sup_{Q \in \mathcal{Q}} P_Q\{\phi_n(\mathbf{Y}_1^n) = 0\}.$$

We consider a universal hypothesis testing problem, also called goodness of fit. It has the following form of  $\mathcal{Q}$ :

$$\mathcal{Q} = \{Q : d(Q, P) \geq \varepsilon\}$$

where  $d$  is a distance function that could change with  $n$ , and  $\varepsilon > 0$ . As discussed in [3], if the distance function is the total variation distance or any distance function dominating the total variation distance, then there is no test that is asymptotically consistent: i.e.  $P_F \rightarrow 0$  and  $P_M \rightarrow 0$  as  $n \rightarrow \infty$ . On the other hand, there is a consistent test if the distance function is the total variation distance defined on a *finite partition* of  $[0, 1]$ : Let

$$\mathbf{A} = \{\mathcal{A}_1, \dots, \mathcal{A}_m\}$$

be a partition of  $[0, 1]$ . The total variation distance defined on this partition is given by

$$d_{\mathbf{A}}(Q, P) = \sup_{\mathcal{A} \subset \mathbf{A}} \{|Q(\mathcal{A}) - P(\mathcal{A})|\}. \quad (1)$$

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Portions of the results presented here were published in abridged form in [1] and [2].

As the number of observations  $n$  increases, it is desirable for a test to not only have a decreasing probability of error, but also be effective against an increasingly larger alternative set  $\mathcal{Q}$ . Therefore, we consider a sequence of distance functions defined with increasingly finer partitions. We restrict ourselves to partitions in which the cells of the partition have equal probabilities under  $P$ :

$$P(A_j) = 1/m \text{ for } 1 \leq j \leq m. \quad (2)$$

One reason to consider uniform cells, as argued in [4], is that the total-variation distance based on this partition gives the best possible distinguishability with respect the Kolmogorov-Smirnov distance: Consider the maximum Kolmogorov-Smirnov distance between the null distribution and any alternative distribution that has zero partition-based total variation distance to the null distribution. Then among any partitions with the same number of cells, the maximum Kolmogorov-Smirnov distance is minimized by the partition with uniform cells.

The dependence between  $n$  and  $m$  plays a significant role on test analysis and synthesis: the *small sample* case in which  $n/m \rightarrow 0$  has a different nature than the *large sample* case in which  $n/m \rightarrow \infty$ . In the large sample case, the number of samples per cell increases to infinity, and thus eventually the underlying probability that  $Y_i$  falls in each cell of  $\mathbf{A}$  can be estimated. This does not hold for the small sample case, in which  $m$  increases faster than  $n$ . The goal of this paper is to find an appropriate analysis criterion for the small sample problem.

#### A. Related work

In this section, we review related results with emphasis on the type of analysis used and the asymptotic settings considered. Many of the results reviewed apply to cases more general than (2).

Examples of partitioned-based tests for small sample problems include Pearson's chi-square test, Generalized Likelihood Ratio Test (GLRT) and the coincidence-based test proposed in [5].

Existing results differ in the asymptotic setting considered, which can be roughly classified into three cases: 1)  $m$  is fixed; 2)  $m$  is increasing and  $m = O(n)$ ; 3)  $n = o(m)$  and  $m = o(n^2)$ . There is no need to consider the case  $n = O(\sqrt{m})$  because the converse result (lower-bounds on probability of error) established in [5] indicates that no asymptotically consistent test exists if  $n = O(\sqrt{m})$ .

There are three predominant types of analysis:

1. Asymptotic consistency / sample complexity analysis: This type of analysis characterizes how fast  $m$  can increase with  $n$ , while still ensuring that  $\limsup_{n \rightarrow \infty} P_F < \delta$ ,  $\limsup_{n \rightarrow \infty} P_M < \delta$  for any small  $\delta \geq 0$ . Finer results on  $P_F$  and  $P_M$  are obtained in Central Limit Theorem (CLT) and large deviations analysis.

2. CLT analysis: CLTs are applied to obtain asymptotic approximations of the distributions of the test statistic under both hypotheses. It is usually assumed that  $\varepsilon \rightarrow 0$  as a function of  $n$ , i.e., the set of alternative distributions becomes closer to the null distribution as  $n$  increases. This ensures that the decision boundary of the test is close to both the null distribution and the alternative distributions, so that the probabilities of false alarm and missed detection can be analyzed using the CLT. Under this choice of  $\varepsilon$ ,  $P_F$  and  $P_M$  usually converge to nonzero values. The results characterize how the limits of  $P_F$  and  $P_M$  differ for different tests.

3. large deviations analysis: The normalized limits (or asymptotic expansions) of  $\log(P_F(\phi))$  and  $\log(P_M(\phi))$  are studied. The distance  $\varepsilon > 0$  is held to be a constant in large deviations analysis. The proper normalization of  $\log(P_F(\phi))$  and  $\log(P_M(\phi))$  must first be identified, and then the normalized limits are calculated.

Consider the case where  $m$  is fixed.

- a) Pearson's chi-square statistic and GLRT statistic are asymptotically distributed as a chi-square distribution whose degree of freedom is  $m - 1$ . These results and their extensions can be found in [6, 7, 8, 9, 10, 11].
- b) The performance of Pearson's chi-square test and GLRT is analyzed in [12] using the large deviations analysis.

The following *error exponent* criterion is used to evaluate a test  $\phi$ :

$$\begin{aligned} I_F(\phi) &:= -\limsup_{n \rightarrow \infty} \frac{1}{n} \log(P_F(\phi_n)), \\ I_M(\phi) &:= -\limsup_{n \rightarrow \infty} \frac{1}{n} \log(P_M(\phi_n)). \end{aligned} \quad (3)$$

The GLRT is shown to have *optimal* error exponents while Pearson's chi-square test does not. Our use of the term error exponent follows [13].

Next consider the case  $m = O(n)$ .

- a) Pearson's chi-square test and GLRT are both asymptotically consistent (For example, see [14]).
- b) Pearson's chi-square statistic and the GLRT statistic both have asymptotically normal distributions. The first work in this line is [15]. Extensions and applications of this result can be found in [16, 17, 18, 19, 20, 21].
- c) A lower-bound on the best achievable probability of error in CLT analysis is given in [14]: Under the condition  $0 < \liminf_{n \rightarrow \infty} \frac{\varepsilon}{\sqrt{m}} \leq \limsup_{n \rightarrow \infty} \frac{\varepsilon}{\sqrt{m}} < \infty$ , Pearson's chi-square test is asymptotically optimal. That is, for any test whose limit of  $P_F$  is no larger than that of Pearson's chi-square test, the limit of its  $P_M$  is asymptotically no smaller than that of Pearson's chi-square test. This result applies to the range of  $m$  satisfying  $m = o(n^2)$ .
- d) An achievability result (A lower-bound on the error exponent) and a complementing converse result (An upper-bound on the error exponent) in the large deviations analysis have been obtained in [3]: There exists a test for which  $P_F$  and  $P_M$  both decay *exponentially* fast with respect to  $n$ , i.e.,  $I_F$  and  $I_M$  defined in (3) are both nonzero, if and only if  $m = O(n)$ . Other large deviations and moderate-deviations analyses of GLRT and Pearson's chi-square test can be found in [22, 23, 24, 25, 26, 27]

Finally consider the small sample case where  $n = o(m)$  and  $m = o(n^2)$ .

- a) Pearson's chi-square test is known to be asymptotically consistent [14]. Two others tests shown to be asymptotically consistent is the test based on counting pairwise-collisions [28] and the coincidence-based test [5]. An approach to extend tests designed for uniform cells (2) to non-uniform cells has been proposed in [29].
- b) Results on the asymptotic distribution of Pearson's chi-square statistic and the GLRT statistic have been obtained in [30, 31].

To the best of our knowledge, the proper normalization for the large deviations analysis has not been identified before in the small sample case.<sup>1</sup> We note that the classical error exponent analysis is not suitable.

## B. Our contributions

The new large deviations framework proposed here is motivated by and analogous to the classical error exponent (3) in the large sample case. While the classical error exponent is defined with the normalization  $n$ , our main results imply that for the small sample problem the following generalized error exponent is best for asymptotic analysis, defined with respect to the normalization  $r(m, n) = n^2/m$ :

$$\begin{aligned} J_F(\phi) &:= -\limsup_{n \rightarrow \infty} \frac{1}{r(m, n)} \log(P_F(\phi_n)), \\ J_M(\phi) &:= -\limsup_{n \rightarrow \infty} \frac{1}{r(m, n)} \log(P_M(\phi_n)). \end{aligned} \quad (4)$$

The generalized error exponents give the following approximation to the probabilities of false alarm and missed detection:

$$P_F \approx e^{-r(n, m)J_F}, \quad P_M \approx e^{-r(n, m)J_M}. \quad (5)$$

The generalized error exponent provides new insights that are not available from asymptotic consistency, or CLT analysis. More precisely, the following results are established:

1. The best achievable probability of error  $P_e = \max\{P_F, P_M\}$ , decays as  $-\log(P_e) = r(n, m)J(1 + o(1))$ , where  $r(n, m) = n^2/m$ . This is applicable not only for the case where the set of alternative distributions is defined by the total variation distance in (1), but also for a broad collection of distance functions.

2. A class of tests based on the separable statistics, including the coincidence-based test  $\phi^*$ , is shown to achieve the *optimal* pair of generalized error exponents  $J_F$  and  $J_M$ :

$$J_M(\phi^*) = \max\{J_M(\phi) : J_F(\phi) \geq J_F(\phi^*)\}.$$

The exact formulae for these generalized error exponents are obtained.

3. The performance of Pearson's chi-square test is asymptotically worse than the optimal test.

<sup>1</sup>Combining the upper-bounds on probability of error given in [5, 29] with the Chernoff inequality gives a loose upper-bound on the asymptotic probability error and does not yield the proper normalization.

### C. Organization of the paper

The paper is organized as follows: The universal hypothesis testing problems and tests are presented in Section II. The main achievability and converse results on generalized error exponents are described in Section III. Extensions of the coincidence-based test are given in Section IV. Performance characterization of Pearson's chi-square test is given in Section V. In Section VI, it is shown that the generalized error exponent criterion is also applicable when the set of alternative distributions is defined using many other distance functions. The paper is concluded in Section VII.

## II. MODELS AND PRELIMINARIES

Here we introduce a more general model based on a sequence of universal hypothesis testing problems, each with a finite number of outcomes (a finite alphabet). Consider an i.i.d. sequence of observations  $\mathbf{Z}_1^n := \{Z_1, \dots, Z_n\}$  where  $Z_i \in [m] := \{1, 2, \dots, m\}$ . Let  $\mathcal{P}_m$  denote the collection of probability mass functions (p.m.f.s) on  $[m]$ . We have two hypotheses: Under the null hypothesis  $H_0$ , the p.m.f. of  $Z_i$  is given by  $p$ , the uniform distribution on  $[m]$ :

$$p_j = 1/m \text{ for } j \in [m]. \quad (6)$$

Under the alternative hypothesis  $H_1$ , the p.m.f. of  $Z_i$  belongs to a set  $\mathcal{Q}_n$  given by

$$\mathcal{Q}_n := \{q \in \mathcal{P}_m : d(q, p) \geq \varepsilon\} \quad (7)$$

where  $d$  is taken to be the total variation distance  $d_{TV}$  defined for any pair of p.m.f.s on  $[m]$ :

$$d_{TV}(q, p) = \sup_{B \subseteq [m]} \{|q(B) - p(B)|\}.$$

A test  $\phi = \{\phi_n\}_{n \geq 1}$  is given by a sequence of binary-valued functions  $\phi_n : [m]^n \rightarrow \{0, 1\}$ . The test decides in favor of  $H_0$  if  $\phi_n(\mathbf{Z}_1^n) = 0$ . The test is required to be powerful against the set  $\mathcal{Q}_n$  of alternative p.m.f.s, and thus its performance is evaluated using the probabilities of false alarm  $P_F(\phi_n)$  and worst-case probability of missed detection  $P_M(\phi_n)$ :

$$\begin{aligned} P_F(\phi_n) &:= \mathbb{P}_p\{\phi_n(\mathbf{Z}_1^n) = 1\}, \\ P_{M,q}(\phi_n) &:= \mathbb{P}_q\{\phi_n(\mathbf{Z}_1^n) = 0\}, \\ P_M(\phi_n) &:= \sup_{q \in \mathcal{Q}_n} \mathbb{P}_q\{\phi_n(\mathbf{Z}_1^n) = 0\}. \end{aligned}$$

An important class of tests is based on the *separable statistics* (see [30]). A separable statistic is a test statistic of the form

$$S_n = \sum_{j=1}^m f_j(n\Gamma_j^n),$$

where

$$\Gamma_j^n := \frac{1}{n} \sum_{i=1}^n \mathbb{I}\{Z_i = j\} \quad (8)$$

is the empirical distribution. General theorems on asymptotic distributions and asymptotic moments of separable statistics are available in [30]. Large deviations analysis for the case  $m = O(n)$  is given in [25, 26]. We are not aware of previous general large deviations results for the small sample case where  $n = o(m)$ .

In this paper, we examine two tests based on separable statistics: Pearson's chi-square test [32] and the coincidence-based test introduced in [5].

After normalization, the test statistic of Pearson's chi-square test is given by

$$S_n^P = \frac{n}{m} \sum_{j=1}^m \frac{(n\Gamma_j^n - np_j)^2}{np_j}. \quad (9)$$

The test is given by  $\phi_n^P(\mathbf{Z}_1^n) = \mathbb{I}\{S_n^P \geq \tau_n\}$ .

The test statistic of the coincidence-based test is given by,

$$S_n^* = - \sum_{j=1}^m \mathbb{I}\{n\Gamma_j^n = 1\}. \quad (10)$$

This test statistic  $S_n^*$  counts the number of symbols in  $[m]$  that appear in the sequence exactly *once*. The coincidence-based test is given by  $\phi_n^*(Z_1) = \mathbb{I}\{S_n^* \geq \mathbb{E}_p[S_n^*] + \tau_n\}$ . The coincidence-based test is applicable only when the null distribution is uniform.

An important difference between  $S_n^*$  and  $S_n^P$  is that  $f_j$  is bounded in  $S_n^*$ , while this is not true in  $S_n^P$ . In Section V, we show that this difference has a significant impact on tests' performance,

#### *Applications to continuously-valued observations*

Tests designed for finite-valued observations can be applied to solve a universal hypothesis testing problem with continuously-valued observations by first partitioning observation space. Consider a measurable space  $(\mathbf{Y}, \mathcal{B})$ , and let  $\mathbf{Y}_1^n = \{Y_1, \dots, Y_n\}$  be an i.i.d. sequence of observations with  $Y_i \in \mathbf{Y}$ . We have two hypotheses:

$$H0 : Y_i \sim P, \quad H1 : Y_i \sim Q \in \mathcal{Q} \quad (11)$$

To apply a test designed for the finite-valued observations, we start with a partition of  $\mathbf{Y}$ :

$$\mathbf{A} = \{A_1, \dots, A_m\}$$

where  $\cup_{1 \leq j \leq m} A_j = \mathbf{Y}$ . The observation  $Y_i$  is mapped to a finite-valued observation via  $\mathcal{T} : \mathbf{Y} \rightarrow [m]$ :  $Z_i := \mathcal{T}(Y_i) = j$  if  $Y_i \in A_j$ . Then a test defined for finite-valued observations can be applied towards  $\{Z_i\}$ . Assume that the partition is chosen so that the marginal of  $Z_i$  is uniform,

$$P(A_j) = \frac{1}{m}. \quad (12)$$

Then tests designed for a uniform null distribution are applicable, such as the coincidence-based test.

This partition-based approach gives tests that are optimal for the model introduced in Section I. More precisely, suppose that the set of alternative distributions is defined as

$$\mathcal{Q} = \{Q : d_{\mathbf{A}}(Q, P) \geq \varepsilon\}$$

where  $d_{\mathbf{A}}$  is defined in (1). Then in terms of the probability of false alarm and worst-case probability of missed detection, without loss of optimality we can restrict our attention to tests whose test statistics take constant value on each cell  $A_j$  of the partition. This is exactly the collection of partition-based tests we have described.

The model introduced in Section I assumes that the alternative distribution  $Q$  is absolutely continuously with respect to  $P$ . The partition-based tests are still applicable when  $Q$  is not absolutely continuous with respect to  $P$ , provided that the tests for finite-valued observations are designed for a more general model where we allow  $p$  not to have full support: Instead of (6), let the null distribution  $p$  be

$$p_j = 1/k \text{ for } 1 \leq j \leq k, p_j = 0 \text{ for } k < j \leq m.$$

The generalized error exponent analysis still applies except the normalization should be  $n^2/k$  instead of  $n^2/m$ .

### III. GENERALIZED ERROR EXPONENTS

In this section, we describe the main results on the proper normalization for large deviations analysis for the small sample universal hypothesis testing problem. The following assumption is imposed throughout:

**Assumption 1.**  $n = o(m)$  and  $m = o(n^2)$ .

To show that the proper normalization to be used in the definition of generalized error exponent is  $n^2/m$ , we need to establish:

- 1) There is a test for which both generalized error exponents are non-zero, and therefore this normalization is not too large.

- 2) For any test, at least one of the generalized error exponents is finite, and therefore this normalization is not too small.

These are established in Theorem 1 and Theorem 2. These two theorems characterize the achievable region of  $(J_F, J_M)$ . This is depicted in Fig. 1. The boundary of the achievable region is given by the following formulae: For  $\tau \in [0, \underline{\kappa}(\varepsilon) - 1]$ ,

$$\begin{aligned} J_F^*(\tau) &:= \sup_{\theta \geq 0} \left\{ \theta \tau - \frac{1}{2} (e^{2\theta} - (1 + 2\theta)) \right\}, \\ J_M^*(\tau) &:= \sup_{\theta \geq 0} \left\{ \theta (\underline{\kappa}(\varepsilon) - 1 - \tau) - \frac{1}{2} (e^{-2\theta} - (1 - 2\theta)) \underline{\kappa}(\varepsilon) \right\}, \end{aligned} \quad (13)$$

where  $\underline{\kappa} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is the  $C^1$  function,

$$\underline{\kappa}(\varepsilon) = \begin{cases} 1 + 4\varepsilon^2, & \varepsilon < 0.5, \\ 1 + \varepsilon/(1 - \varepsilon), & \varepsilon \geq 0.5. \end{cases} \quad (14)$$

**Theorem 1 (Achievability).** *The coincidence-based test  $\phi^*$  achieves the generalized error exponents given in (13), i.e., for any  $\tau \in [0, \underline{\kappa}(\varepsilon) - 1]$ , if the sequence of thresholds  $\{\tau_n\}$  is chosen so that,*

$$\tau = \lim_{n \rightarrow \infty} m\tau_n/n^2, \quad (15)$$

*then the coincidence-based test has the generalized error exponents:*

$$J_F(\phi^*) = J_F^*(\tau), \quad J_M(\phi^*) = J_M^*(\tau). \quad (16)$$

**Theorem 2 (Converse).** *Consider any  $\tau \in [0, \kappa(\varepsilon) - 1]$ . For any test  $\phi$  satisfying*

$$J_F(\phi) \geq J_F^*(\tau),$$

*the following upper-bound on the generalized error exponent of missed detection holds:*

$$J_M(\phi) \leq J_M^*(\tau).$$

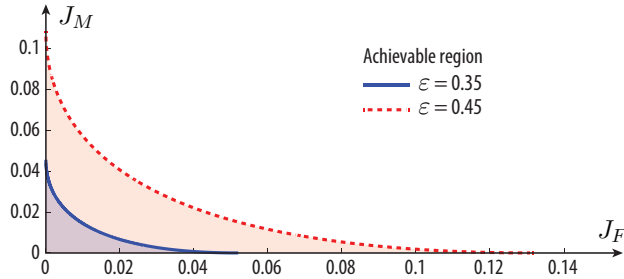


Fig. 1. Achievable region when  $\varepsilon = 0.35$  and  $\varepsilon = 0.45$  given by the lower-bound in Theorem 1 and upper-bound in Theorem 2. The lower and upper bound meet over the entire region.

We now use the approximation in (5) given by the generalized error exponent analysis to the actual empirical performance of the coincidence-based test  $\phi^*$ . The results are shown in Fig. 2 for  $\varepsilon = 0.35$  and Fig. 3 for  $\varepsilon = 0.45$ . We choose the threshold  $\tau$  based on (16) so that  $J_F$  and  $J_M$  are the same. The generalized error exponents are estimates of the *slope* of  $\log(P_F)$  and  $\log(P_M)$  with respect to  $r(n, m)$ . It can be observed that the slope from the theoretical approximation by generalized error exponents approximately matches the slope of the simulated value. The remaining difference between the theoretical and the empirical slope in Fig. 3 is mainly due to two reasons: First, the threshold chosen is based on the first order approximation. It can be observed from the figure that the slope for  $P_M$  is slightly smaller than the predicted slope while the one for  $P_F$  is larger. A slightly larger threshold might yield slopes that are closer to the predicted. Second, the generalized error exponent is only the first term in the asymptotic expansion of  $\log(P_F)$  and  $\log(P_M)$ . Higher order terms might capture the remaining difference.

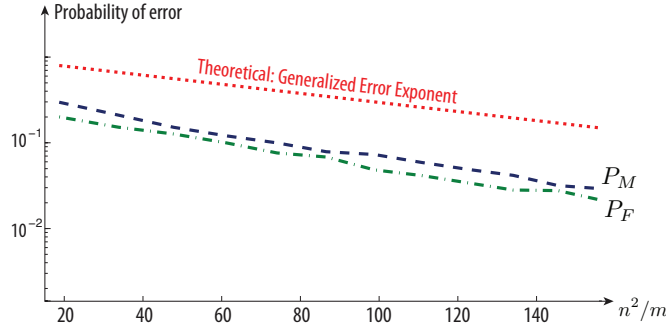


Fig. 2. Performance of  $\phi^*$  with  $\varepsilon = 0.35$ .

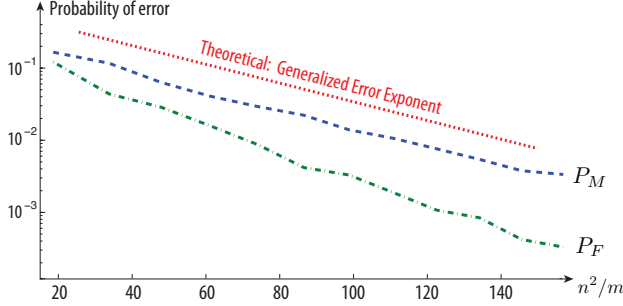


Fig. 3. Performance of  $\phi^*$  with  $\varepsilon = 0.45$ .

#### A. Rate function and worst-case distributions

Similar to the large deviations for the large sample case, we can define a rate function for the small sample case. Consider the coincidence-based test  $\phi^*$ . Consider the following restricted set of alternative distributions:

$$\mathcal{P}_m^b = \{q \in \mathcal{P}_m : \max_j q_j \leq \gamma/m\}, \quad (17)$$

where  $\gamma$  is a large positive constant satisfying  $\gamma \geq \max\{2/(1-\varepsilon), 4\varepsilon\}$ . This restricted set of distributions has bounded likelihood ratios with respect to the uniform distribution  $p$ . The rate function for this test is associated with a sequence of distributions  $\mathbf{q} = \{q^{(1)}, q^{(2)}, q^{(3)}, \dots\}$  with  $q^{(n)} \in \mathcal{P}_m^b$  as follows:

$$J_{\mathbf{q}}(\phi^*, \tau) = -\limsup_{n \rightarrow \infty} \frac{m}{n^2} \log(\mathbb{P}_{q^{(n)}}\{S_n^* \leq \mathbb{E}_p[S_n^*] + \frac{n^2}{m}\tau\}).$$

We show that  $J$  is a function of the following quantity:

$$\kappa(\mathbf{q}) := \liminf_n \sum_j \frac{(q_j^{(n)})^2}{p_j}. \quad (18)$$

#### Theorem 3.

$$J_{\mathbf{q}}(\phi^*, \tau) = \sup_{\theta \geq 0} \{\theta(-1 - \tau) - \frac{1}{2}(e^{-2\theta} - 1)\kappa(\mathbf{q})\}. \quad (19)$$

Its proof is given in Appendix B.

The rate function can be applied to identify the sequence of worst-case alternative distributions, for which the probability of missed detection is asymptotically the largest. Note that  $J_{\mathbf{q}}(\phi^*, \tau)$  is monotonically increasing in  $\kappa(\mathbf{q})$ . Therefore, the smaller the quantity  $\kappa(\mathbf{q})$ , the larger the probability of missed detection associated with  $\mathbf{q}$ . The sequence of distributions achieving the minimum  $\kappa(\mathbf{q})$  is given in the following lemma:

**Lemma 1.** When  $p$  is the uniform distribution, we have

$$\inf_{q \in \mathcal{Q}_n} \left( \sum_{j=1}^m \frac{q_j^2}{p_j} \right) = (1 + \underline{\kappa}(\varepsilon))(1 + o(1)). \quad (20)$$

The infimum is achieved by the following bi-uniform distribution:

1. When  $\varepsilon < 0.5$ ,

$$q_j^* = \begin{cases} 1/m + \varepsilon/\lfloor m/2 \rfloor, & j \leq \lfloor m/2 \rfloor, \\ 1/m - \varepsilon/\lceil m/2 \rceil, & j > \lfloor m/2 \rfloor. \end{cases} \quad (21)$$

2. When  $\varepsilon \geq 0.5$ ,

$$q_j^* = \begin{cases} 1/\lfloor m(1 - \varepsilon) \rfloor, & j \leq \lfloor m(1 - \varepsilon) \rfloor, \\ 0, & j > \lfloor m(1 - \varepsilon) \rfloor. \end{cases} \quad (22)$$

Thus, the worst case distributions are identified as *bi-uniform* distributions whose p.m.f.s only take two possible values.

*Proof of Lemma 1:* The main task is to show that any optimizer  $q^*$  is a bi-uniform distribution. The formulae (21) and (22) follow from solving the optimization in (20) restricted to bi-uniform distributions.

Let  $\mathcal{J}_+ = \{j : q_j^* \geq p_j\}$ ,  $\mathcal{J}_- = \{j : q_j^* < p_j\}$ . The following quadratic programming problem has a unique optimal solution  $x^* = q^*$ :

$$\begin{aligned} \min \quad & \sum_{j \in \mathcal{J}_+} x_j^2, \\ \text{s.t.} \quad & \sum_{j \in \mathcal{J}_+} x_j = \sum_{j \in \mathcal{J}_+} q_j^*, \\ & x_j = q_j^* \text{ for } j \in \mathcal{J}_-, \\ & x_j \geq p_j \text{ for } j \in \mathcal{J}_+. \end{aligned}$$

By Jensen's inequality,  $x^*$  must satisfy  $x_j^* = x_{j'}^*$  for all  $j, j' \in \mathcal{J}_+$ . Thus,  $q^*$  also satisfies  $q_j^* = q_{j'}^*$  for all  $j, j' \in \mathcal{J}_+$ . The same conclusion holds for  $j \in \mathcal{J}_-$ . Consequently,  $q^*$  must be a bi-uniform distribution. ■

### B. Sketch of the proofs for Theorem 1 and Theorem 2

The large deviations characterization of  $P_F$  for the coincidence-based test follows from the following asymptotic approximation of the logarithmic moment generating function of its test statistic:

$$\log(\mathbb{E}_p[\exp\{\theta(n - S_n^*)\}]) = \frac{1}{2} \frac{n^2}{m} \left( m \sum_{j=1}^m p_j^2 \right) (e^{-2\theta} - 1) + O\left(\frac{n^3}{m^2}\right) + O(1).$$

A characterization of  $P_M$  is obtained in similar way except we need to work with the set of alternative distributions. We show that the probability of missed detection is dominated by that associated with the worst-case distributions given in Lemma 1. The details are given in Appendix B.

The main idea to prove the converse result is the following: A sequence of events  $\{B_{n,\tau,\delta}\}$  is constructed so that (i) the probability of these events can be lower-bounded based on the condition on  $P_F$ ; (ii) the probability of missed detection conditioned on these events is lower-bounded. The key to the proof is the following inequality:

$$P_M(\phi_n) \geq \sup_{q \in \mathcal{Q}_n} P_q(\{\phi_n = 0\} \cap B_{n,\tau,\delta}) \geq \sup_{q \in \mathcal{Q}_n} \frac{q^n}{p^n} (\{\phi_n = 0\} \cap B_{n,\tau,\delta}) P_p(\{\phi_n = 0\} \cap B_{n,\tau,\delta}).$$

A lower-bound on the second term follows from the construction of the events and the assumption on the probability of false alarm. To lower-bound the first term, we construct a collection of distributions over which the largest likelihood ratio is always lower-bounded on the event  $B_{n,\tau,\delta}$ . These distributions are obtained by taking the worst-case distribution  $q^*$  given in (21) and permuting the symbols in  $[m]$ . Let  $U_m$  denote the collection of all subsets of  $[m]$  whose cardinality is  $\lfloor m/2 \rfloor$ . For each set  $\mathcal{U} \in U_m$ , define the distribution  $q_{\mathcal{U}}$  as

$$q_{\mathcal{U},j} = \begin{cases} 1/m + \varepsilon/\lfloor m/2 \rfloor, & j \in \mathcal{U}; \\ 1/m - \varepsilon/\lceil m/2 \rceil, & j \in [m] \setminus \mathcal{U}. \end{cases} \quad (23)$$

Then a lower-bound is established for

$$\sup_{\mathcal{U} \in U_m} \frac{q_{\mathcal{U}}^n}{p^n} (\{\phi_n = 0\} \cap B_{n,\tau,\delta}).$$

The details are given in Appendix D.



#### IV. EXTENSIONS OF THE COINCIDENCE-BASED TEST

This section collects together extensions of Section III in terms of tests and models. We first propose a collection of tests that extend the coincidence-based test, and provide the freedom for fine-tuning the performance for finite samples. We then propose an extension of the coincidence-based test for non-uniform  $p$ .

##### A. Extensions considering symbols appearing more than once

The coincidence-based test uses only the number of symbols that appear in the sequence exactly once. We now add terms to the test statistic that also depend on the number of symbols appearing more than once to create a broader collection of tests. Conditions will be established under which these tests have optimal generalized error exponents. Consider the class of test statistics of the following form:

$$S_n^{*+} = S_n^* + \sum_{l=2}^{\bar{l}} v_l \mathbb{I}\{n\Gamma_j^n = l\}. \quad (24)$$

The test is given by

$$\phi^{*+}(\mathbf{Z}_1) = \mathbb{I}\{S_n^{*+} - \mathbb{E}_p[S_n^{*+}] \geq \tau_n\}.$$

**Theorem 4.** *If  $\bar{l} < \infty$ ,  $v_2 = 0$ , and  $v_l \geq 0$  for all  $3 \leq l \leq \bar{l}$ , then the test  $\phi^{*+}$  achieves the optimal generalized error exponents given in (13).*

Its proof is given in Appendix C.

The additional terms for  $l \geq 3$  in the separable statistic give us ways to fine-tune the test for a better finite-sample performance. One interesting question is to obtain finer asymptotic approximations of  $\log(P_F)$  and  $\log(P_M)$  that provide guidance on how to select the weights  $\{v_l\}$ .

For the case with  $v_2 \neq 0$ , we have the following conjecture:

**Conjecture 1.** *If  $S_n^{*+}$  satisfies  $\bar{l} < \infty$ ,  $v_2 > -2$ , and  $v_l \geq 0$  for all  $3 \leq l \leq \bar{l}$ , then the test is optimal in terms of the generalized error exponent.*

##### B. Extensions to non-uniform $p$

The coincidence-based test can be extended to the case where  $p$  is not necessarily uniform but the likelihood ratio between  $p$  and the uniform distribution remains bounded.

**Assumption 2.** *There exists a constant  $\eta > 0$  such that  $\max_j mp_j \leq \eta$  holds for all  $n$ .*

The following separable statistic is considered,

$$S_n^W = \sum_{j=1}^m f_j(n\Gamma_j^n)$$

with

$$f_j(n\Gamma_j^n) = \begin{cases} \frac{1}{2}n^2p_j^2, & n\Gamma_j^n = 0, \\ -np_j, & n\Gamma_j^n = 1, \\ 1, & n\Gamma_j^n = 2, \\ 0, & \text{others.} \end{cases} \quad (25)$$

The *weighted coincidence-based test* is given by  $\phi_n^W = \mathbb{I}\{S_n^W \geq \tau_n\}$ .

The choice of coefficients given in (25) ensures  $\mathbb{E}_\nu[S_n^W]$  approximates the  $\ell_2$ -distance between  $\nu$  and  $p$ :

**Lemma 2.** *For  $\nu \in \mathcal{P}_m^b$ , the expectation of  $S_n^W$  is given by:*

$$\mathbb{E}_\nu[S_n^W] = \frac{1}{2} \frac{n^2}{m} \left[ m \sum_{j=1}^m (\nu_j - p_j)^2 \right] + O\left(\frac{n^3}{m^2}\right).$$

The proposed test has nonzero generalized error exponents:

**Theorem 5.** Suppose Assumption 1 and Assumption 2 hold. For  $\tau \in (0, 2\varepsilon^2)$  where  $\tau$  is defined in (15), the test  $\phi^W$  has nonzero generalized error exponents:

$$J_F(\phi^W) > 0, \quad J_M(\phi^W) > 0.$$

Its proof is given in Appendix C.

## V. PEARSON'S CHI-SQUARE TEST

In this section, we investigate the performance of Pearson's chi-square test given in (9). We find that this test has a zero generalized error exponent, and therefore its probability of error is asymptotically larger than that of the coincidence-based test.

Pearson's chi-square test is asymptotically consistent in the small sample case:

**Proposition 1** (Asymptotic consistency). *Under Assumption 1, there exists a sequence of thresholds  $\{\tau_n\}$ , with which the Pearson's chi-square test is asymptotically consistent:*

$$\lim_{n \rightarrow \infty} P_F(\phi_n^P) = 0, \quad \lim_{n \rightarrow \infty} P_M(\phi_n^P) = 0.$$

We give a proof that highlights the relationship between Pearson's chi-square test and the coincidence-based test.

*Proof of Proposition 1:* Let  $\tau_n = n + \frac{1}{2} \frac{n^2}{m} (\kappa(\varepsilon) - 1)$ . Applying approximations of moments of separable statistic given in Lemma 6 and Lemma 8, we obtain

$$\begin{aligned} \mathbb{E}_p[S_n^P] &= n + O\left(\frac{n^3}{m^2}\right), \\ \text{Var}_p[S_n^P] &= 2 \frac{n^2}{m} \left(m \sum_{j=1}^m p_j^2\right) (1 + o(1)). \end{aligned} \tag{26}$$

Applying Chebyshev's inequality gives  $\lim_{n \rightarrow \infty} P_F(\phi_n^P) = 0$ .

We bound  $P_M(\phi_n^P)$  by coupling Pearson's chi-square statistic  $S_n^P$  with the coincidence-based test statistic  $S_n^*$ :

$$S_n^P = \sum_{j=1}^m (n\Gamma_j^n - np_j)^2 = \sum_{j=1}^m (n\Gamma_j^n)^2 - \frac{n^2}{m} \geq 2 \sum_{j=1}^n \mathbb{I}\{n\Gamma_j^n \geq 2\} n\Gamma_j^n + \sum_{j=1}^m \mathbb{I}\{n\Gamma_j^n = 1\} - \frac{n^2}{m} = 2n + S_n^* - \frac{n^2}{m},$$

where the inequality follows from  $(n\Gamma_j^n)^2 \geq 2(n\Gamma_j^n)$  when  $n\Gamma_j^n > 1$ . Consequently,

$$\{S_n^P \leq \tau_n\} \subseteq \{S_n^* \leq \tau_n - 2n + \frac{n^2}{m}\}. \tag{27}$$

The asymptotic approximation on the expectation of  $S_n^*$  obtained from Lemma 6 gives

$$\tau_n - 2n + \frac{n^2}{m} = \mathbb{E}_p[S_n^*] + \frac{1}{2} \frac{n^2}{m} (\kappa(\varepsilon) - 1) + O\left(\frac{n^3}{m^2}\right).$$

It follows from Theorem 1 that the coincidence-based test is asymptotically consistent. Thus

$$\lim_{n \rightarrow \infty} \sup_{q \in \mathcal{Q}_n} P_q\{S_n^* \leq \tau_n - 2n + \frac{n^2}{m}\} = 0.$$

Applying (27), we obtain

$$\lim_{n \rightarrow \infty} \sup_{q \in \mathcal{Q}_n} P_q\{S_n^P \leq \tau_n\} = 0.$$

■

However, the probability of false alarm of Pearson's chi-square test is asymptotically larger than that the coincidence-based test: We show that its generalized error exponent of false alarm is zero:

**Theorem 6.** Suppose Assumption 1 hold. Assume in addition that  $m = o(n^2 / \log(n)^2)$ . If the sequence of thresholds is chosen so that

$$\lim_{n \rightarrow \infty} P_M(\phi_n^P) = 0, \tag{28}$$

then the generalized error exponent of false alarm is zero, i.e.,

$$J_F(\phi^P) = 0. \quad (29)$$

We conjecture that the conclusion holds without the assumption  $m = o(n^2/\log(n)^2)$ .

We now compare Pearson's chi-square test and the coincidence-based test. Note that Pearson's chi-square test statistic can be written as

$$S_n^P = -\frac{n^2}{m} + \sum_{j=1}^m \mathbb{I}\{n\Gamma_j^n = 1\} + \sum_{j=1}^m 4\mathbb{I}\{n\Gamma_j^n = 2\} + \sum_{l=3}^{\infty} \sum_{j=1}^m l^2 \mathbb{I}\{n\Gamma_j^n = l\}. \quad (30)$$

The main difference between these two tests are how the coefficients of  $\mathbb{I}\{n\Gamma_j^n = l\}$  for  $l \geq 2$  are chosen: Remove all the terms corresponding to  $l \geq 3$  and consider the following separable statistic:

$$S_n^{P0} = -\frac{n^2}{m} + \sum_{j=1}^m \mathbb{I}\{n\Gamma_j^n = 1\} + \sum_{j=1}^m 4\mathbb{I}\{n\Gamma_j^n = 2\}. \quad (31)$$

Then we have the following relationship between these three test statistics:

$$\Omega^P := \{S_n^P \leq \check{\tau}_n\} \subset \Omega^* := \{S_n^* \leq \tau_n\} \subset \Omega^{P0} := \{S_n^{P0} \leq \check{\tau}_n\} \quad (32)$$

where the thresholds  $\tau_n$  and  $\check{\tau}_n$  satisfy  $\check{\tau}_n = \tau_n + 2n - \frac{n^2}{m}$ . This is depicted in Fig. 4. Note that the region which Pearson's chi-square test decides in favor of  $H1$  is larger than the coincidence-based test, and the probability that the empirical distribution fall into this region is asymptotically larger than  $\exp\{-\alpha n^2/m\}$  for any  $\alpha > 0$ . This is made precise in the proof of Theorem 6. On the other hand, we can show that the test associated with  $\phi^{P0}$  has  $J_M = 0$  by considering a sequence of alternative distributions whose likelihood ratios with respect to  $p$  increase to infinity. In sum, we have

- 1)  $J_F(\phi^P) = 0, J_M(\phi^P) > 0$ ;
- 2)  $J_F(\phi^*) > 0, J_M(\phi^*) > 0$ ;
- 3)  $J_F(\phi^{P0}) > 0, J_M(\phi^{P0}) = 0$ .

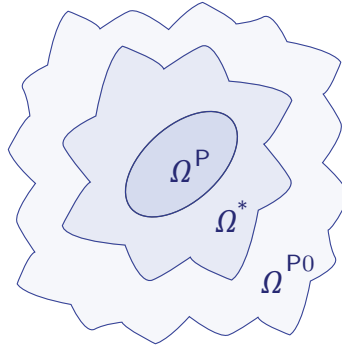


Fig. 4. Decision regions in the space of p.m.f. for Pearson's chi-square test, the coincidence-based test and the test given in (31).

*Proof of Theorem 6:* The requirement  $P_M(\phi_n^P) \rightarrow 0$  imposes an upper-bound on the threshold  $\tau_n$  for  $\phi^P$ :

**Lemma 3.** In order for (28) to hold, for large enough  $n$ , we must have

$$\tau_n \leq \bar{\tau}_n := \mathbb{E}_p[S_n^P] + \frac{n^2}{m} \underline{\kappa}(\varepsilon) + 2\frac{n}{\sqrt{m}}.$$

Consider the event that the first symbol appears many times:

$$A_n := \{n\Gamma_1^n = \lfloor \frac{n\sqrt{2\underline{\kappa}(\varepsilon)}}{\sqrt{m}} \rfloor\}.$$

In the event  $A_n$ , the first term  $f_1(n\Gamma_1^n)$  in the summation in the definition of  $S_n^P$  given in (9) is approximately  $2\frac{n^2}{m}\underline{\kappa}(\varepsilon)$ . This drives the value of  $S_n^P$  above the threshold  $\tau_n$ . Thus the probability of false alarm conditioned on

this event converges to *one*, as summarized in Lemma 4. On the other hand, the probability of  $A_n$  does not decay exponentially fast with respect to  $n^2/m$ , as summarized in Lemma 5.

**Lemma 4.**

$$\mathbb{P}_p\{S_n^P \geq \bar{\tau}_n | A_n\} = 1 - o(1).$$

**Lemma 5.**

$$-\lim_{n \rightarrow \infty} \frac{m}{n^2} \log(\mathbb{P}_p\{A_n\}) = 0.$$

Combining Lemma 3, Lemma 4 and Lemma 5 together, we conclude

$$J_F(\phi^P) \leq -\liminf_{n \rightarrow \infty} \frac{m}{n^2} \log(\mathbb{P}_p\{S_n^P \geq \bar{\tau}_n | A_n\} \mathbb{P}_p\{A_n\}) = 0.$$

The proofs of these three lemmas are given in Appendix E. ■

## VI. ALTERNATIVE DISTRIBUTIONS BASED ON $f$ -DIVERGENCE

The set of alternative distributions studied in previous sections is defined using the total variation distance. The generalized error exponent analysis with the same normalization  $r(n, m) = n^2/m$  also applies to other distance functions, as we show in Proposition 2 and Proposition 3. In this section, the set of alternative distributions  $\mathcal{Q}_n$  defined in (7) is based on a general distance function  $d$  rather than  $d = d_{TV}$ . Examples include the KL divergence

$$d_{KL}(q, p) = \sum_j q_j \log(q_j/p_j),$$

and its generalization known as  $f$ -divergence,

$$d_f(q, p) = \sum_j p_j f(q_j/p_j), \tag{33}$$

where  $f$  is a convex function with  $f(1) = 0$ .

Conditions under which the generalized error exponent analysis applies are contained in the following:

**Proposition 2.** *Suppose the distance function  $d$  satisfies*

- 1)  $d(q, p) \geq \alpha d_{TV}(q, p)$  for some  $\alpha > 0$ .
- 2)

$$\liminf_k \inf \left\{ \sum_j \frac{q_j^2}{p_j} : d(q, p) \geq \varepsilon, q \in \mathcal{P}_m \right\} > 0.$$

*Then  $n^2/m$  is the appropriate normalization for the large deviations analysis for small  $\varepsilon > 0$ : There exists a test  $\phi$  such that*

$$J_F(\phi) > 0, J_M(\phi) > 0.$$

*There is a constant  $\bar{J}$  satisfying  $0 < \bar{J} < \infty$  such that for any test  $\phi$ , we have*

$$\min\{J_F(\phi), J_M(\phi)\} \leq \bar{J}.$$

For the set of alternative distributions defined in (7) with the  $f$ -divergence  $d = d_f$ , the generalized error exponent can be applied subject to conditions on  $f$ :

**Proposition 3.** *Suppose  $f$  satisfies the following conditions:*

- 1) For some  $0 < x < 1$ ,

$$\frac{1}{2}(f(1-x) + f(1+x)) > f(1).$$

- 2) There is a constant  $\alpha > 0$  such that for all  $x$ ,

$$f(x) \leq \alpha(x-1)^2.$$

*Then  $n^2/m$  is the appropriate normalization for the large deviations analysis for small  $\varepsilon > 0$ : There exists a test  $\phi$  such that*

$$J_F(\phi) > 0, J_M(\phi) > 0.$$

There is a constant  $\bar{J}$  satisfying  $0 < \bar{J} < \infty$  such that for any test  $\phi$ , we have

$$\min\{J_F(\phi), J_M(\phi)\} \leq \bar{J}.$$

*Proof of Proposition 2:* The converse result in Theorem 2 is proved by showing that the worst-case probability of missed detection over the set of distributions given in (23) is lower-bounded regardless of the test used. The first condition in Proposition 2 guarantees that these distributions are still in the set  $\mathcal{Q}_n$  of alternative distributions.

For the achievability result, the critical step is to show that the rate function is positive for any alternative distribution whose likelihood ratio with respect to  $p$  is bounded. The second condition in Proposition 2 guarantees that  $\kappa$  defined in (18) is positive, which in turn ensures a positive rate function. ■

*Proof of Proposition 3:* The proof is similar to that of Proposition 2. The first condition of Proposition 3 ensures that the collection of bi-uniform distributions given in (23) used in the proof of the converse result is in the set  $\mathcal{Q}_n$  of alternative distributions: For  $q_U$  defined in (23) with  $\varepsilon$  replaced by  $\varepsilon'$ , for even  $m$ , for small enough  $\varepsilon$ , we have

$$d_f(q_U, p) = \frac{1}{2}f(1 + 2\varepsilon') + \frac{1}{2}f(1 - 2\varepsilon') \geq \varepsilon$$

The second condition implies that

$$\alpha \sum_j \frac{q_j^2}{p_j} \geq d_f(q, p) \geq \varepsilon.$$

Thus, the rate function is positive for any alternative distribution whose likelihood ratio with respect to  $p$  is bounded. ■

## VII. CONCLUSIONS AND DISCUSSIONS

We have shown that the classical error exponent criterion, which appears in the large deviation analysis for universal hypothesis testing problems with large number of samples, can be extended to the small sample case, provided the normalization is modified to account for both the sample size  $n$  and the alphabet size  $m$ .

We offer a few discussions on the results and point out directions for future research:

1. The analysis in this paper is of asymptotic nature. The generalized error exponent gives the leading term in the asymptotic expansion of the probability of error. Finer approximations are valuable especially for characterizing the finite sample performance when  $n/m$  is not very small. For example, finer approximations can reveal the difference among the class of tests described in Section IV-A that has the same generalized error exponents.

2. The size of alphabet  $m$ , is used in this and previous work to capture the “complexity” of the hypothesis testing problem in the case where the null distribution is uniform. It remains to see how this can be generalized to other cases, where the null distribution is far from uniform or has a countably infinite support. A possible generalization of the size of alphabet is the Rényi entropy of  $p$ , which is equal to  $\log(m)$  when  $p$  is uniform.

3. It is desirable to establish general large deviation characterizations of separable statistics for small sample problems, similar to those established for  $n \asymp m$  in [25, 26]. Such results could provide more insights on how the coefficients of a separable statistic affect the test’s performance. For example, how the performance of a test with test statistic  $\sum_{j=1}^m (n\Gamma_j^n - np_j)^\rho$  varies with  $\rho$ ?

4. We have focused on the simple goodness-of-fit problem in this paper, in which  $p$  is fully specified. A natural extension is the composite goodness-of-fit problem in which  $p$  is not fully specified but assumed to be in a known set. A similar generalized error exponent concept should exist for the composite case.

5. There are many other problems for which the approach presented in this paper is relevant. Examples include the classification problem [33, 34, 35], the problem of testing whether two distributions are close [36, 37], and probability estimation over a large or unknown alphabet [38, 39, 40].

In the recent work [41] it is shown how to adapt the methods presented to the classification problem. The generalized error exponent analysis is applied to characterize the different ways in which the number of training samples and the number of test samples affect the performance of classification algorithms.

6. Topological structure often contains critical information that is easily ignored in the approaches focused on in this work. In particular, in this paper we have not considered any notion of distance between points in the alphabet. Other approaches such as the support vector machine, or more recent work such as [42] are based primarily on topology. It will be desirable to create a coherent bridge between the approach developed here and

topological approaches to hypothesis testing. It is likely that current information-theoretic tools can help to create these bridges, such as concepts from lossy source-coding. We are also considering extensions of the work described here to the feature selection problem of [43, 44] in which  $m$  is interpreted as the number of features rather than the alphabet size.

## ORGANIZATION OF THE APPENDIX

Approximations to the moments of the separable statistics are given in Appendix A, and the results are used in the rest of the proofs.

The proofs of Theorem 1 and Theorem 3 are given in Appendix B. Similar arguments are used in the proofs of Theorem 4 and Theorem 5 given in Appendix C.

The proof of Theorem 2 given in Appendix D can be read almost independently of Appendix B and C.

The lemmas supporting the proof of Theorem 6 are given in Appendix E, and can be read independently of Appendix B, C and D.

## APPENDIX A MOMENTS OF SEPARABLE STATISTICS

This section provides a survey of results on asymptotic approximations to moments of separable statistics. The results hold for the distributions in the set  $\mathcal{P}_m^b$  defined in (17).

**Lemma 6** (Expectation of a separable statistic). *Consider the separable statistic  $\sum_{j=1}^m f_j(n\Gamma_j^n)$ . Suppose we have  $\max_j |f_j(x)| \leq a_0 e^{a_0 x}$  for some  $a_0 > 0$ . Then its expectation for  $\nu \in \mathcal{P}_m^b$  is given by:*

$$\mathbb{E}_\nu\left[\sum_{j=1}^m f_j(n\Gamma_j^n)\right] = \sum_j f_j(0) + n \sum_{j=1}^m \nu_j(f_j(1) - f_j(0)) + \frac{1}{2} \frac{n^2}{m} \left(m \sum_{j=1}^m \nu_j^2\right) (f_j(0) - 2f_j(1) + f_j(2)) + O\left(\frac{n^3}{m^2}\right).$$

*Proof:* We have  $\nu_j^3 \binom{n}{3} |f_j(3)| = O\left(\frac{n^3}{m^3}\right)$  and

$$\sum_{x=4}^{\infty} \nu_j^x \binom{n}{x} |f_j(x)| \leq a_0 \sum_{x=4}^{\infty} \left(\frac{e^{a_0} \gamma n}{m}\right)^x \leq \frac{a_0}{|\log(e^{a_0} \gamma n/m)|} \left(\frac{e^{a_0} \gamma n}{m}\right)^3 = O\left(\frac{n^3}{m^3}\right).$$

Consequently,

$$\begin{aligned} & \mathbb{E}_\nu\left[\sum_{j=1}^m f_j(n\Gamma_j^n)\right] \\ &= \sum_{j=1}^m [f_j(0)(1-\nu_j)^n + f_j(1)n\nu_j(1-\nu_j)^{n-1} \\ & \quad + f_j(2)\binom{n}{2}\nu_j^2(1-\nu_j)^{n-2} + O\left(\frac{n^3}{m^3}\right)] \\ &= \sum_j f_j(0) + n \sum_{j=1}^m \nu_j(f_j(1) - f_j(0)) \\ & \quad + \frac{n^2}{2} \sum_{j=1}^m \nu_j^2(f_j(0) - 2f_j(1) + f_j(2)) + O\left(\frac{n^3}{m^2}\right). \end{aligned}$$

■

Lemma 6 leads to Lemma 2, as well as the following asymptotic approximation of the expectation of  $S_n^*$ :

**Lemma 7.** *For any  $\nu \in \mathcal{P}_m^b$ :*

$$\mathbb{E}_\nu[S_n^*] = -n + \frac{n^2}{m} \left(m \sum_{j=1}^m \nu_j^2\right) + O\left(\frac{n^3}{m^2}\right).$$

This will be used in the proof of Theorem 1.

**Lemma 8** (Variance of a separable statistic). *Consider a symmetric separable statistic  $\sum_{j=1}^m f(n\Gamma_j^n)$ . Suppose  $|f(x)| \leq a_0 e^{a_0 x}$  for some  $a_0 > 0$ . Moreover, suppose  $f(0) = 0$  and  $f(2) \neq 2f(1)$ . Then its variance for  $\nu \in \mathcal{P}_m^b$  is given by*

$$\text{Var}_\nu \left[ \sum_{j=1}^m f(n\Gamma_j^n) \right] = \frac{1}{2} \frac{n^2}{m} (f(2) - 2f(1))^2 \left( m \sum_{j=1}^m \nu_j^2 \right) (1 + o(1)).$$

Lemma 8 is the combination of Equation 2.11 and Equation 2.20 in [30].

## APPENDIX B PROOFS OF THEOREM 1 AND THEOREM 3

The proof of Theorem 1 and Theorem 3 is based on the Chernoff bound and the Gärtner-Ellis Theorem. The key step is to obtain an asymptotic approximation to the logarithmic moment generating function of the test statistic. To simplify the presentation, instead of  $S_n^*$  we work with the following statistic:

$$\tilde{S}_n^* = \sum_{j=1}^m \mathbb{I}\{n\Gamma_j^n = 1\} - n.$$

Its logarithmic moment generating is denoted by

$$\Lambda_{\nu, \tilde{S}_n^*}(\theta) := \log(\mathbb{E}_\nu[\exp\{\theta \tilde{S}_n^*\}]). \quad (34)$$

Asymptotic approximations or bounds to  $\Lambda_{\nu, \tilde{S}_n^*}(\theta)$  are presented in the next two sections.

### A. Approximation to the logarithmic moment generating function for distributions in $\mathcal{P}_m^b$

Bounds and approximations for  $\Lambda_{\nu, \tilde{S}_n^*}$  are first obtained for the restricted set of distributions  $\mathcal{P}_m^b$  defined in (17).

**Proposition 4.** *For any  $\nu \in \mathcal{P}_m^b$ , the logarithmic moment generating function for the statistic  $\tilde{S}_n^*$  has the following asymptotic expansion*

$$\Lambda_{\nu, \tilde{S}_n^*}(\theta) = \frac{1}{2} \frac{n^2}{m} \left( m \sum_{j=1}^m \nu_j^2 \right) (e^{-2\theta} - 1) + O\left(\frac{n^3}{m^2}\right) + O(1). \quad (35)$$

The approximation errors  $O(\frac{n^3}{m^2})$  and  $O(1)$  are uniform over the set  $\mathcal{P}_m^b$ .

The proof uses the Poissonization technique, and the procedure is applicable for many separable statistics including  $S_n^*$ : Let  $\{X_j\}$  be a sequence of independent Poisson random variables with parameter  $\lambda \nu_j$  for some  $\lambda > 0$ . Then for any integers  $u_1, \dots, u_m$  satisfying  $\sum_{j=1}^m u_j = n$ , we have

$$\mathbb{P}\{n\Gamma_j^n = u_j, \text{ for all } j\} = \mathbb{P}\{X_j = u_j, \text{ for all } j \mid \sum_{j=1}^m X_j = n\}.$$

Therefore, the moment generating function of a separable statistic  $\sum_{j=1}^m f_j(n\Gamma_j^n)$  admits the following representation:

$$\mathbb{E}_\nu[\exp\{\theta \sum_{j=1}^m f_j(n\Gamma_j^n)\}] = \mathbb{E}[\exp\{\theta \sum_{j=1}^m f_j(X_j)\} \mid \sum_{j=1}^m X_j = n].$$

It is related to the moment generating function  $A_\lambda(\theta)$  for  $\sum_{j=1}^m f_j(X_j)$  as follows:

$$\begin{aligned} A_\lambda(\theta) &:= \mathbb{E}[\exp\{\theta \sum_{j=1}^m f_j(X_j)\}] \\ &= \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} e^{-\lambda} \mathbb{E}[\exp\{\theta \sum_{j=1}^m f_j(X_j)\} \mid \sum_{j=1}^m X_j = n] \\ &= \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} e^{-\lambda} \mathbb{E}_\nu[\exp\{\theta \sum_{j=1}^m f_j(n\Gamma_j^n)\}]. \end{aligned}$$

It follows from the independence of the variables  $\{X_j\}$  that the moment generating function  $A_\lambda(\theta)$  has the following formula:

$$A_\lambda(\theta) = \prod_{j=1}^m \left( \sum_{k=0}^{\infty} \frac{(\lambda \nu_j)^k}{k!} e^{-\lambda \nu_j} e^{\theta f_j(k)} \right).$$

Since  $A_\lambda(\theta)$  is analytic in  $\lambda$ , the moment generating function of  $\sum_{j=1}^m f_j(n\Gamma_j^n)$  can be obtained via Cauchy's theorem:

$$\mathbb{E}_\nu[\exp\{\theta \sum_{j=1}^m f_j(n\Gamma_j^n)\}] = \frac{n!}{2\pi} \oint e^\lambda A_\lambda(\theta) \frac{d\lambda}{\lambda^{n+1}}, \quad (36)$$

where the integration is carried out along any closed contour around  $\lambda = 0$  in the complex plane. These arguments lead to the following lemma:

**Lemma 9.** *The moment generating function of the separable statistic  $\sum_{j=1}^m f_j(n\Gamma_j^n)$  is given by*

$$\begin{aligned} \mathbb{E}_\nu[\exp\{\theta \sum_{j=1}^m f_j(n\Gamma_j^n)\}] \\ = \frac{n!}{2\pi} \oint e^\lambda \prod_{j=1}^m \left( \sum_{k=0}^{\infty} \frac{(\lambda \nu_j)^k}{k!} e^{-\lambda \nu_j} e^{\theta f_j(k)} \right) \frac{d\lambda}{\lambda^{n+1}}. \end{aligned}$$

*Proof of Proposition 4:* Applying Lemma 9 with  $f_j(1) = 1, f_j(k) = 0$  for  $k \neq 1$ , we obtain

$$\mathbb{E}_\nu[\exp\{\theta(\tilde{S}_n^*)\}] = e^{-\theta n} \frac{n!}{2\pi i} \oint g(\lambda) d\lambda \quad (37)$$

where

$$g(\lambda) = e^\lambda \prod_{j=1}^m (1 - (\lambda \nu_j) e^{-\lambda \nu_j} + (\lambda \nu_j) e^{-\lambda \nu_j} e^\theta) \frac{1}{\lambda^{n+1}}.$$

The rest of the proof is an application of the saddle point method [45]. It consists of two steps: The first step is to pick a particular contour around  $\lambda = 0$  to carry out the integration. It is desirable to have a contour along which  $g(\lambda)$  behaves violently:  $g(\lambda)$  is large on a small interval on the contour and significantly smaller at the rest, so that the value of integral can be approximated by integrating over this small interval. Such a contour can be found, by identifying a *saddle point* of  $g(\lambda)$  at which the derivative of  $g(\lambda)$  vanishes, and then pick a contour that goes through the saddle point. The second step is to apply the Laplace method to estimate the integral along the contour.

We now apply the first step of the saddle point method: identifying the saddle point and defining the contour for integration. Note that the derivative of  $g$  is given by

$$\frac{d}{d\lambda} g(\lambda) = g(\lambda) \left[ \sum_{j=1}^m \frac{\nu_j (e^\theta - 1 + e^{\lambda \nu_j})}{\lambda \nu_j (e^\theta - 1) + e^{\lambda \nu_j}} - \frac{n+1}{\lambda} \right].$$

To simplify the derivation, we select a point that is close to a saddle point, defined as the solution to

$$\sum_{j=1}^m \frac{\lambda \nu_j (e^\theta - 1 + e^{\lambda \nu_j})}{\lambda \nu_j (e^\theta - 1) + e^{\lambda \nu_j}} = n. \quad (38)$$

If  $\lambda$  on the left-hand side was taken to be a saddle point, then the right-hand side would be  $n+1$  instead of  $n$ , and we will see this error is negligible for our purposes.

Equation (38) has one unique real-valued nonnegative solution, which we denote by  $\lambda_0$ . To see this, note that when restricting  $\lambda$  to  $[0, \infty)$ , the left-hand-side is a continuous and strictly increasing function of  $\lambda$ . Moreover, its value is 0 when  $\lambda = 0$ , increases to  $\infty$  when  $\lambda$  increases to  $\infty$ .

We now obtain an asymptotic expansion of  $\lambda_0$ . We first show that  $\lambda_0 = O(n)$ . When  $\theta \geq 0$ , using the fact that  $0 \leq x e^{-x} \leq e^{-1}$  and  $0 \leq e^{-x} \leq 1$  for  $x \geq 0$ , we obtain

$$\frac{1}{1 + e^{-1}(e^\theta - 1)} \leq \frac{e^\theta - 1 + e^{\lambda \nu_j}}{\lambda \nu_j (e^\theta - 1) + e^{\lambda \nu_j}} \leq e^\theta.$$



Substituting this into (38) leads to

$$ne^{-\theta} \leq \lambda_0 \leq n(1 + e^{-1}(e^\theta - 1)). \quad (39)$$

When  $\theta < 0$ , we obtain

$$e^\theta \leq \frac{e^\theta - 1 + e^{\lambda\nu_j}}{\lambda\nu_j(e^\theta - 1) + e^{\lambda\nu_j}} \leq \frac{1}{1 + e^{-1}(e^\theta - 1)}.$$

Substituting this into (38) leads to

$$n(1 + e^{-1}(e^\theta - 1)) \leq \lambda_0 \leq ne^{-\theta}. \quad (40)$$

It follows from the bounds (39), (40) and  $\nu \in \mathcal{P}_m^b$  that  $\lambda_0\nu_j = o(1)$ . Thus the denominator of (38) satisfies

$$\lambda_0\nu_j(e^\theta - 1) + e^{\lambda_0\nu_j} = 1 + o(1).$$

Substituting this into (38) leads to

$$\sum_{j=1}^m \lambda_0\nu_j(e^\theta - 1 + e^{\lambda_0\nu_j}) = n(1 + o(1)).$$

Consequently,

$$\lambda_0 = ne^{-\theta}(1 + o(1)).$$

To obtain a refined approximation, let  $w := \lambda_0 e^\theta / n - 1$ . Consequently,

$$\lambda_0 = ne^{-\theta}(1 + w). \quad (41)$$

An approximation for  $w$  will be obtained: Since  $\lambda_0\nu_j = O(\frac{n}{m})$ , we have that the numerator and denominator in the summand of (38) satisfy

$$\begin{aligned} \lambda_0\nu_j(e^\theta - 1 + e^{\lambda_0\nu_j}) &= \lambda_0\nu_j(e^\theta + \lambda_0\nu_j + O(\frac{n^2}{m^2})), \\ \lambda_0\nu_j(e^\theta - 1) + e^{\lambda_0\nu_j} &= 1 + \lambda_0\nu_j e^\theta + O(\frac{n^2}{m^2}). \end{aligned}$$

Thus,

$$\begin{aligned} &\sum_{j=1}^m \frac{\lambda_0\nu_j(e^\theta - 1 + e^{\lambda_0\nu_j})}{\lambda_0\nu_j(e^\theta - 1) + e^{\lambda_0\nu_j}} \\ &= \sum_j [\lambda_0\nu_j e^\theta + \lambda_0^2\nu_j^2(1 - e^{2\theta}) + O(\frac{n^3}{m^3})]. \end{aligned}$$

Substituting this and (41) into (38) leads to

$$w + n \sum_j \nu_j^2(1 + w)^2(e^{-2\theta} - 1) = O(\frac{n^2}{m^3}),$$

which gives

$$w = n \sum_j \nu_j^2(1 - e^{-2\theta})(1 + O(\frac{n}{m})) = O(\frac{n}{m}). \quad (42)$$

The integration in (37) is now carried out along the closed contour given by  $\lambda = \lambda_0 e^{i\psi} = ne^{-\theta}(1 + w)e^{i\psi}$ :

$$\begin{aligned} \mathbb{E}_\nu[\exp\{\theta(\tilde{S}_n^*)\}] &= e^{-\theta n} \frac{n!}{2\pi} \int_{-\pi}^{\pi} g(\lambda_0 e^{i\psi}) \lambda_0 e^{i\psi} d\psi \\ &= \frac{n!}{2\pi} \lambda_0^{-n} e^{-\theta n} \operatorname{Re} \left[ \int_{-\pi}^{\pi} h(\psi) d\psi \right]. \end{aligned} \quad (43)$$

where

$$h(\psi) := e^{-in\psi} \prod_{j=1}^m (\lambda_0\nu_j(e^\theta - 1)e^{i\psi} + e^{\lambda_0\nu_j e^{i\psi}}). \quad (44)$$

We now apply the second step of the saddle point method: estimating the integral by the Laplace method. We begin with a rough estimate of  $h(\psi)$ . It follows from  $\lambda_0 = n^{-\theta}(1 + o(1))$  that

$$\begin{aligned}
h(\psi) &= e^{-in\psi} \prod_{j=1}^m (\lambda_0 \nu_j (e^\theta - 1) e^{i\psi} + 1 + \lambda_0 \nu_j e^{i\psi} + O(\frac{n^2}{m^2})) \\
&= e^{-in\psi} \prod_{j=1}^m (1 + \lambda_0 \nu_j e^\theta e^{i\psi} + O(\frac{n^2}{m^2})) \\
&= e^{-in\psi} \exp\left\{\sum_{j=1}^m (\lambda_0 \nu_j e^\theta e^{i\psi} + O(\frac{n^2}{m^2}))\right\} \\
&= e^{-in\psi} e^n \exp\{-n(1 - e^{i\psi}) + O(\frac{n^2}{m})\}.
\end{aligned} \tag{45}$$

Therefore, for any  $\psi \neq 0$ ,  $|h(\psi)|$  is exponentially smaller than the value of  $h(\psi)$  at  $\psi = 0$ . This suggests that the integral in (43) can be approximated by integrating over a small interval around  $\psi = 0$ . Split the integral in (43) into three parts:

$$\begin{aligned}
I_1 &= \operatorname{Re} \left[ \int_{-\pi/3}^{\pi/3} h(\psi) d\psi \right], \\
I_2 &= \operatorname{Re} \left[ \int_{-\pi}^{-\pi/3} h(\psi) d\psi \right], \\
I_3 &= \operatorname{Re} \left[ \int_{\pi/3}^{\pi} h(\psi) d\psi \right].
\end{aligned} \tag{46}$$

We first estimate  $I_1$ . Denote  $H(\psi) = \log(h(\psi))$ . Simple calculus gives

$$\begin{aligned}
H(\psi) &= -in\psi + \sum_{j=1}^m \log(\lambda_0 \nu_j (e^\theta - 1) e^{i\psi} + \exp\{\lambda_0 \nu_j e^{i\psi}\}), \\
H'(\psi) &= -in + i \sum_{j=1}^m \frac{\lambda_0 \nu_j (e^\theta - 1) e^{i\psi} + \lambda_0 \nu_j e^{i\psi} \exp\{\lambda_0 \nu_j e^{i\psi}\}}{\lambda_0 \nu_j (e^\theta - 1) e^{i\psi} + \exp\{\lambda_0 \nu_j e^{i\psi}\}}, \\
H''(\psi) &= - \sum_{j=1}^m \exp\{\lambda_0 \nu_j e^{i\psi}\} \frac{1}{(\lambda_0 \nu_j (e^\theta - 1) e^{i\psi} + \exp\{\lambda_0 \nu_j e^{i\psi}\})^2} \\
&\quad \times (\lambda_0 \nu_j (e^\theta - 1) e^{i\psi} (1 - \lambda_0 \nu_j e^{i\psi} + \lambda_0^2 \nu_j^2 e^{2i\psi}) + \lambda_0 \nu_j e^{i\psi} \exp\{\lambda_0 \nu_j e^{i\psi}\}).
\end{aligned} \tag{47}$$

It is clear that  $\operatorname{Im}(H(0)) = 0$ . It follows from (38) that  $H'(0) = 0$ . Estimates of  $\operatorname{Re}(H(0))$  and  $H''(\psi)$  are obtained from substituting (41) and (42) into the expression of  $H(\psi)$  and  $H''(\psi)$  and applying asymptotic analysis. In sum,

$$\begin{aligned}
\operatorname{Im}(H(0)) &= 0, \\
\operatorname{Re}(H(0)) &= n(1 + w) - \frac{1}{2}n^2 \left( \sum_{j=1}^m \nu_j^2 \right) (1 - e^{-2\theta}) + O(\frac{n^3}{m^2}), \\
H'(0) &= 0, H''(\psi) = -ne^{i\psi} + O(\frac{n^2}{m}).
\end{aligned} \tag{48}$$

To obtain an upper-bound on  $I_1$ , note that for large enough  $n$  and for any  $\psi \in [-\pi/3, \pi/3]$ , we have  $\operatorname{Re}(H''(\psi)) \leq -0.4n$ . It then follows from the mean value theorem that

$$\operatorname{Re}(H(\psi)) \leq H(0) - 0.2n\psi^2.$$

Consequently, for large enough  $n$  and  $m$ ,

$$I_1 \leq e^{H(0)} \int_{-\pi/3}^{\pi/3} e^{-0.2n\psi^2} d\psi \leq e^{H(0)} \int_{-\infty}^{\infty} e^{-0.2n\psi^2} d\psi = e^{H(0)} \frac{\sqrt{\pi}}{\sqrt{0.2n}}. \tag{49}$$

To obtain a lower-bound on  $I_1$ , we begin with an bound on  $\text{Im}(H''(\psi))$ : Since  $\text{Im}(H''(\psi)) = -n \sin(\psi) + O(\frac{n^2}{m})$ , applying  $|\sin(\psi)| \leq |\psi|$ , we have that for large enough  $n$ , for any  $\psi \in [-\pi/3, \pi/3]$ ,  $|\text{Im}(H''(\psi))| \leq 1.1n|\psi|$ . It also follows from (48) that  $\text{Re}(H''(\psi)) \geq -1.1n$ . Applying the mean value theorem, we conclude that there exists some  $c > 0$  such that for  $\psi \in [-\pi/3, \pi/3]$ ,

$$\begin{aligned} \text{Re}(H(\psi)) &\geq H(0) - 1.1n\psi^2, \\ |\text{Im}(H(\psi))| &\leq 1.1n|\psi|^3 + c\frac{n^2}{m}\psi^2. \end{aligned}$$

Use the short-hand notation  $t_n = 0.1 \min\{n^{-1/3}, \sqrt{m}/(\sqrt{cn})\}$ . For  $\psi \in [-t_n, t_n]$ , we have  $\cos(\text{Im}(H(\psi))) \geq 0.5$ , and thus  $\text{Re}(e^{H(\psi)}) \geq 0.5e^{\text{Re}(H(\psi))}$ . The integration for  $I_1$  is further split into three parts:

$$I_1 = \text{Re}\left[\int_{-\pi/3}^{-t_n} e^{H(\psi)} d\psi\right] + \text{Re}\left[\int_{t_n}^{\pi/3} e^{H(\psi)} d\psi\right] + \text{Re}\left[\int_{-t_n}^{t_n} e^{H(\psi)} d\psi\right].$$

The absolute value of the first term is upper-bounded as follows:

$$\begin{aligned} \left|\int_{-\pi/3}^{-t_n} e^{H(\psi)} d\psi\right| &\leq e^{H(0)} \int_{-\infty}^{-t_n} e^{-0.2n\psi^2} d\psi = t_n e^{H(0)} \int_{-\infty}^{-1} e^{-0.2nt_n^2 \bar{\psi}^2} d\bar{\psi} \\ &\leq t_n e^{H(0)} \int_{-\infty}^{-1} e^{-0.2nt_n^2 |\bar{\psi}|} d\bar{\psi} = e^{H(0)} O\left(\frac{1}{nt_n}\right) = e^{H(0)} o\left(\frac{1}{\sqrt{n}}\right). \end{aligned} \quad (50)$$

The second term is bounded in a similar way. The third term is lower-bounded as follows:

$$\begin{aligned} \text{Re}\left[\int_{-t_n}^{t_n} e^{H(\psi)} d\psi\right] &\geq \int_{-t_n}^{t_n} 0.5e^{\text{Re}(H(\psi))} d\psi \geq 0.5e^{H(0)} \int_{-t_n}^{t_n} e^{-1.1n\psi^2} d\psi \\ &\geq 0.5e^{H(0)} \left[\int_{-\infty}^{\infty} e^{-1.1n\psi^2} d\psi - 2 \int_{-\infty}^{-t_n} e^{-1.1n\psi^2} d\psi\right] \\ &\geq 0.5e^{H(0)} \left(\frac{\sqrt{\pi}}{\sqrt{1.1n}} + O\left(\frac{1}{nt_n}\right)\right) = 0.5e^{H(0)} \frac{\sqrt{\pi}}{\sqrt{1.1n}} (1 + o(1)). \end{aligned}$$

where the last inequality follows from an argument similar to (50). Combining these bounds together, we obtain

$$\begin{aligned} I_1 &\geq \text{Re}\left[\int_{-t_n}^{t_n} e^{H(\psi)} d\psi\right] - |\text{Re}\left[\int_{-\pi/3}^{-t_n} e^{H(\psi)} d\psi\right]| - |\text{Re}\left[\int_{t_n}^{\pi/3} e^{H(\psi)} d\psi\right]| \\ &\geq e^{H(0)} \frac{0.5\sqrt{\pi}}{\sqrt{1.1n}} (1 + o(1)). \end{aligned}$$

Combing this and (49) leads to,

$$I_1 = e^{H(0)} \frac{1}{\sqrt{n}} e^{O(1)} = e^{n(1+o(1))} \frac{1}{\sqrt{n}} e^{O(1)}. \quad (51)$$

where the last equality follows from the estimate of  $H(0)$  given in (48) and (42).

We now estimate  $I_2$  and  $I_3$ . For  $\psi \in [-\pi, -\pi/3] \cup [\pi/3, \pi]$ , we obtain from (45) that  $|h(\psi)| \leq \exp\{0.5n + O(\frac{n^2}{m})\}$ , which implies  $\text{Re}[I_2] + \text{Re}[I_3] = O(e^{0.6n})$ . This shows that  $I_2$  and  $I_3$  are much smaller than  $I_1$ . Thus, the integral

in (43) can be approximated by the estimate of  $I_1$ : Substituting (51) and (48) into (43), we obtain

$$\begin{aligned}
& \mathbb{E}_\nu[\exp\{\theta(\tilde{S}_n^*)\}] \\
&= \frac{n!}{2\pi} \lambda_0^{-n} e^{-\theta n} I_1(1 + o(1)) \\
&= \frac{n!}{2\pi} \lambda_0^{-n} e^{-\theta n} e^{H(0)} \frac{1}{\sqrt{n}} e^{O(1)} (1 + o(1)) \\
&= \frac{n!}{n^n \sqrt{2\pi n}} \left(1 + n \sum_j \nu_j^2 (1 - e^{-2\theta}) + O\left(\frac{n^2}{m^2}\right)\right)^{-n} \\
&\quad \times \exp\left\{\frac{1}{2}n^2 \left(\sum_{j=1}^m \nu_j^2\right) (1 - e^{-2\theta}) + O\left(\frac{n^3}{m^2}\right)\right\} e^{O(1)} \\
&= \frac{n! e^n}{n^n \sqrt{2\pi n}} \exp\left\{-\frac{1}{2}n^2 \left(\sum_{j=1}^m \nu_j^2\right) (1 - e^{-2\theta}) + O\left(\frac{n^3}{m^2}\right)\right\} e^{O(1)}.
\end{aligned}$$

Stirling formula gives  $\frac{n! e^n}{n^n \sqrt{2\pi n}} = 1 + O(\frac{1}{n})$ . The claim of the proposition is obtained on taking logarithm on both sides.  $\blacksquare$

#### B. Approximation to the logarithmic moment generating function for distributions not in $\mathcal{P}_m^b$

We also need to consider distributions in  $\mathcal{Q}_n \setminus \mathcal{P}_m^b$ . For any  $q \in \mathcal{Q}_n \setminus \mathcal{P}_m^b$ , the set of indices  $\mathcal{S}_0 := \{j \in [m] : q_j \geq \gamma m^{-1}\}$  is non-empty. Now fix a small constant  $\eta > 0$ , and consider each index  $j$  in  $\mathcal{S}_0$  in two separate cases, according to whether  $nq_j \geq \eta$ . Denote

$$\mathcal{W}_\eta(q) = \{j : nq_j \geq \eta\}, \quad \beta(q) = \sum_{j \in \mathcal{W}_\eta(q)} q_j.$$

Proposition 5 below addresses the case where  $\beta(q)$  is large. It implies that the probability of missed detection associated with such a distribution is much smaller than that associated with the worst-case distributions: The probability decays exponentially fast with respect to  $n$ , which is larger than  $n^2/m$ . Proposition 6 considers the alternate case, and shows that if  $\beta(q)$  is not large, then a bound similar to that in Proposition 4 holds.

**Proposition 5.** *For all sufficiently small  $\eta > 0$ , any  $\theta \in (0, 0.5]$ , and any  $\underline{\beta} > 0$ , there exists  $n_0$  such that for any  $n > n_0$ , and any  $\nu$  satisfying  $\beta(\nu) \geq \underline{\beta}$ , the following holds,*

$$\Lambda_{\nu, \tilde{S}_n^*}(\theta) \leq -\beta(\nu) \alpha(\theta) n,$$

where  $\alpha(\theta) > 0$ .

**Proposition 6.** *For any  $\delta > 0$ ,  $\theta \in (0, 0.5]$ ,  $\bar{\eta} > 0$ , there exist  $\eta \in (0, \bar{\eta})$ ,  $\bar{\beta} > 0$ , and  $n_0$  such that for any  $n > n_0$ , and any  $\nu$  satisfying  $\beta(\nu) \leq \bar{\beta}$ , the following holds,*

$$\Lambda_{\nu, \tilde{S}_n^*}(\theta) \leq \frac{1}{2} \frac{n^2}{m} \left(m \sum_{j \notin \mathcal{W}_\eta(\nu)} q_j^2\right) (e^{-2\theta} - 1)(1 - \delta).$$

The proofs of Proposition 5 and Proposition 6 use steps similar to those leading to the upper-bound in Proposition 4. However, the approximation given by (41) and (42) is no longer valid, so a different approximation is required. The conclusions on the existence and uniqueness of the solution  $\lambda_0$  and the bounds in (39) are still valid, and our proof begins from there.

To simplify the presentation, we use the following notation similar to the small “ $o$ ” notation: We write  $x = o^\eta(1)$  whenever there exists a function  $s(\eta)$  that does not depend on  $\theta$ ,  $n$ , and  $\nu$ , such that  $|x| \leq s(\eta)$  and  $\lim_{\eta \rightarrow 0} s(\eta) = 0$ .

Consider any  $\eta$  and  $\nu$ . Write  $\mathcal{W}_\eta = \mathcal{W}_\eta(\nu)$ . For any  $j \notin \mathcal{W}_\eta$ , we obtain the expansion of the summand in (38) via the mean value theorem:

$$\frac{\lambda_0 \nu_j (e^\theta - 1 + e^{\lambda_0 \nu_j})}{\lambda_0 \nu_j (e^\theta - 1) + e^{\lambda_0 \nu_j}} = \lambda_0 \nu_j e^\theta + \lambda_0^2 \nu_j^2 (1 - e^{2\theta}) (1 + o^\eta(1)).$$

For any  $j \in \mathcal{W}_\eta$ , the following equality holds:

$$\frac{\lambda_0 \nu_j (e^\theta - 1 + e^{\lambda_0 \nu_j})}{\lambda_0 \nu_j (e^\theta - 1) + e^{\lambda_0 \nu_j}} = D_j \lambda_0 \nu_j e^\theta,$$

where

$$D_j := \frac{e^{-\theta} + e^{-\lambda_0 \nu_j} (1 - e^{-\theta})}{1 + \lambda_0 \nu_j e^{-\lambda_0 \nu_j} (e^\theta - 1)} \geq e^{-2\theta}. \quad (52)$$

Substituting these estimates into (38) leads to

$$\lambda_0 (1 + \sum_{j \in \mathcal{W}_\eta} \nu_j (D_j - 1)) e^\theta + \lambda_0^2 \sum_{j \notin \mathcal{W}_\eta} \nu_j^2 (1 - e^{2\theta}) (1 + o^\eta(1)) = n. \quad (53)$$

Applying  $\lambda_0 \sum_{j \notin \mathcal{W}_\eta} \nu_j^2 \leq \eta \sum_{j \notin \mathcal{W}_\eta} \nu_j \leq \eta$  gives,

$$\lambda_0 = \frac{n e^{-\theta}}{1 + \sum_{j \in \mathcal{W}_\eta} \nu_j (D_j - 1)} (1 + o^\eta(1)).$$

Introducing a variable  $w$  as before,

$$\lambda_0 = \frac{n e^{-\theta}}{1 + \sum_{j \in \mathcal{W}_\eta} \nu_j (D_j - 1)} (1 + w). \quad (54)$$

On substituting (54) into (53), we obtain

$$w = \frac{n (\sum_{j \notin \mathcal{W}_\eta} \nu_j^2 (1 - e^{-2\theta}))}{(1 + \sum_{j \in \mathcal{W}_\eta} \nu_j (D_j - 1))^2} (1 + o^\eta(1)) = o^\eta(1). \quad (55)$$

In the proofs of both propositions, we integrate (37) along the closed contour corresponding to  $\lambda = \lambda_0 e^{i\psi}$  from  $\psi = -\pi$  to  $\psi = \pi$ , and use the same definition of  $h(\psi)$  given in (44) and  $H(\psi) = \log(h(\psi))$ . The integral is given in (43) and our task is to estimate it. We now give the details.

*Proof of Proposition 5:* We first show that any  $\psi$ ,

$$\operatorname{Re}(H(\psi)) \leq H(0) = \sum_j [\lambda_0 \nu_j + \log(1 + \lambda_0 \nu_j e^{-\lambda_0 \nu_j} (e^\theta - 1))]. \quad (56)$$

Thus to bound the integral in (43), we only need to bound  $H(0)$ . For  $\psi \in [-\frac{1}{2}\pi, \frac{1}{2}\pi]$ , the summand in the expression of  $\operatorname{Re}(H(\psi))$  given in (47) is bounded as follows:

$$\begin{aligned} & \operatorname{Re}[\log(\lambda_0 \nu_j (e^\theta - 1) e^{i\psi} + e^{\lambda_0 \nu_j e^{i\psi}})] \\ &= \operatorname{Re}[\log(e^{\lambda_0 \nu_j e^{i\psi}}) + \log(1 + \lambda_0 \nu_j (e^\theta - 1) e^{i\psi} e^{-\lambda_0 \nu_j e^{i\psi}})] \\ &\leq \lambda_0 \nu_j \cos \psi + \log(1 + \lambda_0 \nu_j e^{-\lambda_0 \nu_j \cos \psi} (e^\theta - 1)). \end{aligned} \quad (57)$$

The right-hand side is a convex function of  $\cos \psi$  for  $\psi \in [-\frac{1}{2}\pi, \frac{1}{2}\pi]$ . Thus, it achieves its maximum value at  $\cos \psi = 1$  or  $\cos \psi = 0$ . Note that its value at  $\cos \psi = 1$  is exactly equal to the summand in  $H(0)$ . Moreover, we can show that its value at  $\cos \psi = 1$  is no smaller than its value at  $\cos \psi = 0$ :

$$\begin{aligned} & \lambda_0 \nu_j + \log(1 + \lambda_0 \nu_j (e^\theta - 1) e^{-\lambda_0 \nu_j}) - \log(1 + \lambda_0 \nu_j (e^\theta - 1)) \\ &= \lambda_0 \nu_j + \log\left(\frac{1 + \lambda_0 \nu_j (e^\theta - 1) e^{-\lambda_0 \nu_j}}{1 + \lambda_0 \nu_j (e^\theta - 1)}\right) \leq \lambda_0 \nu_j + \log(e^{-\lambda_0 \nu_j}) = 0, \end{aligned}$$

where the inequality follows from  $\theta \geq 0$ . This leads to (56) for  $\psi \in [-\frac{1}{2}\pi, \frac{1}{2}\pi]$ .

For  $\psi \in [-\pi, -\frac{1}{2}\pi] \cup [\frac{1}{2}\pi, \pi]$ , we have  $|e^{\lambda_0 \nu_j e^{i\psi}}| \leq 1$ . Consequently,

$$|\lambda_0 \nu_j (e^\theta - 1) e^{i\psi} + e^{\lambda_0 \nu_j e^{i\psi}}| \leq 1 + \lambda_0 \nu_j (e^\theta - 1),$$

which leads to

$$\operatorname{Re}[\log(\lambda_0 \nu_j (e^\theta - 1) e^{i\psi} + e^{\lambda_0 \nu_j e^{i\psi}})] \leq \log(1 + \lambda_0 \nu_j (e^\theta - 1)). \quad (58)$$

The right-hand side of (58) is equal to the value of the right-hand side of (57) at  $\cos \psi = 0$ , which has been shown in the previous paragraph to be smaller than  $H(0)$ . This leads to (56) for  $\psi \in [-\pi, -\frac{1}{2}\pi] \cup [\frac{1}{2}\pi, \pi]$ .

We now approximate the right-hand side of (56): For  $j \notin \mathcal{W}_\eta$ , we have

$$\lambda_0 \nu_j + \log(1 + \lambda_0 \nu_j e^{-\lambda_0 \nu_j} (e^\theta - 1)) = \lambda_0 \nu_j^\theta + \frac{1}{2} \lambda_0^2 \nu_j^2 (1 - e^{2\theta})(1 + o^\eta(1)).$$

For  $j \in \mathcal{W}_\eta$ , we have the inequality

$$\lambda_0 \nu_j + \log(1 + \lambda_0 \nu_j e^{-\lambda_0 \nu_j} (e^\theta - 1)) \leq \lambda_0 \nu_j e^\theta + \lambda_0 \nu_j (1 - e^{-\lambda_0 \nu_j})(1 - e^\theta).$$

Substituting these two estimates, and (54), (56) into (43) leads to

$$\begin{aligned} \mathbb{E}_\nu[\exp\{\theta(\tilde{S}_n^*)\}] &\leq \frac{n!}{2\pi} \lambda_0^{-n} e^{-\theta n} \exp\{H(0)\} \\ &\leq n! \lambda_0^{-n} e^{-\theta n} \exp\left\{\sum_{j \notin \mathcal{W}_\eta} [\lambda_0 \nu_j e^\theta + \frac{1}{2} \lambda_0^2 \nu_j^2 (1 - e^{2\theta})(1 + o^\eta(1))]\right\} \\ &\quad \times \exp\left\{\sum_{j \in \mathcal{W}_\eta} [\lambda_0 \nu_j e^\theta + \lambda_0 \nu_j (1 - e^{-\lambda_0 \nu_j})(1 - e^\theta)]\right\} \\ &= \frac{n! e^n}{n^n} \left(1 + \sum_{j \in \mathcal{W}_\eta} \nu_j (D_j - 1)\right)^n (1 + w)^{-n} \\ &\quad \times \exp\left\{-\frac{\frac{1}{2} n^2 \sum_{j \notin \mathcal{W}_\eta} \nu_j^2 (1 - e^{-2\theta})(1 + o^\eta(1))}{1 + \sum_{j \in \mathcal{W}_\eta} \nu_j (D_j - 1)}\right\} \\ &\quad \times \exp\left\{n \left[\frac{(1 + w) + \sum_{j \in \mathcal{W}_\eta} \nu_j (1 - e^{-\lambda_0 \nu_j})(e^{-\theta} - 1)}{1 + \sum_{j \in \mathcal{W}_\eta} \nu_j (D_j - 1)}\right]\right\} \\ &\leq \frac{n! e^n}{n^n} \exp\left\{-n \log(1 + w) + \frac{nw}{1 + \sum_{j \in \mathcal{W}_\eta} \nu_j (D_j - 1)}\right\} \\ &\quad \times \exp\left\{-\frac{\frac{1}{2} n^2 \sum_{j \notin \mathcal{W}_\eta} \nu_j^2 (1 - e^{-2\theta})(1 + o^\eta(1))}{1 + \sum_{j \in \mathcal{W}_\eta} \nu_j (D_j - 1)}\right\} \\ &\quad \times \exp\left\{n \left[\sum_{j \in \mathcal{W}_\eta} \nu_j (D_j - 1) - 1 + \frac{1 + \sum_{j \in \mathcal{W}_\eta} \nu_j (1 - e^{-\lambda_0 \nu_j})(e^{-\theta} - 1)}{1 + \sum_{j \in \mathcal{W}_\eta} \nu_j (D_j - 1)}\right]\right\}. \end{aligned} \tag{59}$$

We now bound each exponential term on the right-hand side of (60). Applying (55) and the lower-bound on  $D_j$  in (52) gives the following bound on the second term:

$$-\frac{\frac{1}{2} n^2 \sum_{j \notin \mathcal{W}_\eta} \nu_j^2 (1 - e^{-2\theta})}{1 + \sum_{j \in \mathcal{W}_\eta} \nu_j (D_j - 1)} \leq -\frac{1}{2} e^{-2\theta} n w (1 + o^\eta(1)). \tag{61}$$

The first exponential term satisfies

$$-n \log(1 + w) + \frac{nw}{1 + \sum_{j \in \mathcal{W}_\eta} \nu_j (D_j - 1)} = -n w o^\eta(1), \tag{62}$$

which follows from (52) and  $w = o^\eta(1)$ . Combining (61) and (62) implies that for small enough  $\eta$ , the sum of the first and second term is *negative*.

The exponent in the last term on the right-hand side of (60) is bounded as follows:

$$\begin{aligned}
& \sum_{j \in \mathcal{W}_\eta} \nu_j(D_j - 1) - 1 + \frac{1 + \sum_{j \in \mathcal{W}_\eta} \nu_j(1 - e^{-\lambda_0 \nu_j})(e^{-\theta} - 1)}{1 + \sum_{j \in \mathcal{W}_\eta} \nu_j(D_j - 1)} \\
&= \frac{(\sum_{j \in \mathcal{W}_\eta} \nu_j(D_j - 1))^2 + \sum_{j \in \mathcal{W}_\eta} \nu_j(1 - e^{-\lambda_0 \nu_j})(e^{-\theta} - 1)}{1 + \sum_{j \in \mathcal{W}_\eta} \nu_j(D_j - 1)} \\
&\leq \frac{(\sum_{j \in \mathcal{W}_\eta} \nu_j) \sum_{j \in \mathcal{W}_\eta} \nu_j(D_j - 1)^2 + \sum_{j \in \mathcal{W}_\eta} \nu_j(1 - e^{-\lambda_0 \nu_j})(e^{-\theta} - 1)}{1 + \sum_{j \in \mathcal{W}_\eta} \nu_j(D_j - 1)} \\
&\leq \frac{\sum_{j \in \mathcal{W}_\eta} \nu_j[(D_j - 1)^2 + (1 - e^{-\lambda_0 \nu_j})(e^{-\theta} - 1)]}{1 + \sum_{j \in \mathcal{W}_\eta} \nu_j(D_j - 1)}
\end{aligned} \tag{63}$$

where the first inequality follows from Jensen's inequality and the second follows from  $\sum_{j \in \mathcal{W}_\eta} \nu_j \leq 1$ .

We first bound the summand in the numerator on the right-hand side of (63). Consider any  $j \in \mathcal{W}_\eta$ . Let  $x := \lambda_0 \nu_j$ . Applying the formula of  $D_j$  in (52) gives

$$(D_j - 1)^2 + (1 - e^{-x})(e^{-\theta} - 1) = \frac{e^{-x} + e^{-\theta} - e^{-x}e^{-\theta}}{(1 + xe^{-x}(e^\theta - 1))^2} [(1 - e^{-x})(e^{-\theta} - 1) + (xe^{-x}(e^\theta - 1))^2]. \tag{64}$$

Let  $t(x) = (1 - e^{-x})(e^{-\theta} - 1) + (xe^{-x}(e^\theta - 1))^2$ . Note that  $j \in \mathcal{W}_\eta$  implies  $n\nu_j \geq \eta$ , which combined with (39) implies  $x = \lambda_0 \nu_j \geq \eta e^{-\theta}$ . Since for  $\theta \in (0, 0.5]$ ,  $t(x)$  is strictly decreasing on  $[0, \infty)$ , we obtain  $t(x) \leq t(\eta e^{-\theta}) < 0$ . Substituting this into (64) and using the elementary fact that

$$\frac{e^{-x} + e^{-\theta} - e^{-x}e^{-\theta}}{(1 + xe^{-x}(e^\theta - 1))^2} \leq e^{-3\theta},$$

we obtain

$$(D_j - 1)^2 + (1 - e^{-x})(e^{-\theta} - 1) \leq -e^{-3\theta} t(\eta e^{-\theta}).$$

The denominator of on the right-hand side of (63) is positive and upper-bounded by 1 because  $D_j \leq 1$ . Combining the bounds on the numerator and denominator gives a bound on the exponent in the last term on the right-hand side of (60)

$$\sum_{j \in \mathcal{W}_\eta} \nu_j(D_j - 1) - 1 + \frac{(1 + \sum_{j \in \mathcal{W}_\eta} \nu_j(1 - e^{-\lambda_0 \nu_j})(e^{-\theta} - 1))}{1 + \sum_{j \in \mathcal{W}_\eta} \nu_j(D_j - 1)} \leq -\beta(\nu)\alpha(\theta) \leq 0, \tag{65}$$

where

$$\alpha(\theta) = \frac{1}{3} e^{-3\theta} [(1 - e^{-\eta e^{-\theta}})(e^{-\theta} - 1) + (\eta e^{-\theta} e^{-\eta e^{-\theta}}(e^\theta - 1))^2].$$

Combining (61), (62) and (65) and using the fact that the right-hand sides of (61) (62) are negative, we obtain:

$$\mathbb{E}_\nu[\exp\{\theta(\tilde{S}_n^*)\}] \leq \frac{n!e^n}{\sqrt{2\pi n}n^n} \sqrt{2\pi n} \exp\{-n\beta(\nu)\alpha(\theta)\}.$$

Taking the logarithm on both side and applying Stirling's formula leads to

$$\Lambda_{\nu, S_n^*}(\theta) \leq -n\beta(\nu)\alpha(\theta) + \frac{1}{2} \log(2\pi n) + O\left(\frac{1}{n}\right).$$

Since  $\beta(\nu) \geq \underline{\beta}$ , the second term  $\frac{1}{2} \log(2\pi n)$  becomes negligible comparing to the first term for large  $n$ . This leads to the claim of the proposition.  $\blacksquare$

*Proof of Proposition 6:* We pick  $\bar{\beta}$  so that  $\bar{\beta} = o^\eta(1)$ . It then follows that

$$\sum_{j \in \mathcal{W}_\eta} \nu_j(D_j - 1) = o^\eta(1) \tag{66}$$

Substituting this into (54) and (55) gives

$$\lambda_0 = ne^{-\theta}(1 + o^\eta(1)), \quad w = n\left(\sum_{j \notin \mathcal{W}_\eta} \nu_j^2\right)(1 - e^{-2\theta})(1 + o^\eta(1)). \quad (67)$$

The rest of the proof is similar to the proof of Proposition 4. Applying (56) to  $j \in \mathcal{W}_\eta$ , we obtain

$$\begin{aligned} |h(\psi)| &\leq |e^{-in\psi} \prod_{j \notin \mathcal{W}_\eta} (\lambda_0 \nu_j (e^\theta - 1) e^{i\psi} + e^{\lambda_0 \nu_j e^{i\psi}})| \prod_{j \in \mathcal{W}_\eta} \exp\{\lambda_0 \nu_j + \log(1 + \lambda_0 \nu_j e^{-\lambda_0 \nu_j} (e^\theta - 1))\} \\ &\leq |e^{-in\psi}| \exp\left\{\left(\sum_{j \notin \mathcal{W}_\eta} \lambda_0 \nu_j e^\theta \cos \psi (1 + o^\eta(1))\right) + \sum_{j \in \mathcal{W}_\eta} \lambda_0 \nu_j e^\theta\right\} \\ &= e^n \exp\{-n(1 - \cos \psi + o^\eta(1))\}. \end{aligned} \quad (68)$$

It is clear from (68) that the integrand is large at the interval around 0. Thus, we again split the integral in (43) into three parts  $I_1$ ,  $I_2$  and  $I_3$  as in (46). We will show later that  $I_2$  and  $I_3$  are much smaller than  $I_1$ .

We first upper-bound  $I_1$ . Similar to (48), we have

$$\operatorname{Im}(H(0)) = 0, \operatorname{Re}(H'(0)) = 0, \operatorname{Im}(H'(0)) = 0.$$

We now estimate  $H''(\psi)$ , whose exactly formula is given in (47). Consider  $j \in \mathcal{W}_\eta$ . For  $\psi \in [-\pi/3, \pi/3]$ , we have the following inequality:

$$\begin{aligned} |1 + \lambda_0 \nu_j (e^\theta - 1) e^{i\psi} \exp\{-\lambda_0 \nu_j e^{i\psi}\}| &\geq 1, \\ |\lambda_0 \nu_j (e^\theta - 1) e^{i\psi} (1 - \lambda_0 \nu_j e^{i\psi} + \lambda_0^2 \nu_j^2 e^{2i\psi}) \exp\{-\lambda_0 \nu_j e^{i\psi}\} + \lambda_0 \nu_j e^{i\psi}| &\leq 100 \lambda_0 \nu_j e^\theta. \end{aligned}$$

Substituting these into (47), we obtain  $|H''(\psi)| \leq 100 \bar{\beta} n (1 + o^\eta(1)) = n o^\eta(1)$ . Substituting this and the estimate (67) into the expression of  $H''(\psi)$  leads to

$$H''(\psi) = -n(e^{i\psi} + o^\eta(1)).$$

Note that the assumption of the proposition allows us to take very small  $\eta$ . We choose it small enough so that the term  $o^\eta(1)$  in the above equation is smaller than 0.05. Then for large enough  $n$ , for any  $\psi \in [-\pi/3, \pi/3]$ , we have  $\operatorname{Re}(H''(\psi)) \leq -0.4n$ . It follows from the mean value theorem that

$$\operatorname{Re}(H(\psi)) \leq H(0) - 0.2n\psi^2.$$

Consequently, for large enough  $n$  and  $m$ , we have

$$I_1 \leq e^{H(0)} \int_{-\pi/3}^{\pi/3} e^{-0.4\psi^2} d\psi \leq e^{H(0)} \int_{-\infty}^{\infty} e^{-0.4\psi^2} d\psi = e^{H(0)} \frac{\sqrt{\pi}}{\sqrt{0.4n}}. \quad (69)$$

We now bound the tails  $I_2$  and  $I_3$ . For  $\psi \in [-\pi, -\pi/3] \cup [\pi/3, \pi]$ , we obtain from (68) that  $|h(\psi)| \leq \exp\{0.5n(1 + o^\eta(1))\}$ . Thus, for small enough  $\eta$ , we have

$$\operatorname{Re}[I_2] + \operatorname{Re}[I_3] = O(e^{0.6n}).$$

Substituting the estimate for  $I_1$ ,  $I_2$  and  $I_3$  into (43) gives

$$\mathbb{E}_\nu[\exp\{\theta(\tilde{S}_n^*)\}] \leq \frac{n!}{\sqrt{1.6n\pi}} \lambda_0^{-n} e^{-\theta n} e^{H(0)} (1 + o(1)).$$

Note that the right-hand side is almost the same as the right-hand side of (59) except for the multiplication term  $\frac{1}{\sqrt{1.6n\pi}}(1 + o(1))$ . Thus, we can bound it using the right-hand side of (60) after taking into account this additional multiplication term. We obtain

$$\mathbb{E}_\nu[\exp\{\theta(\tilde{S}_n^*)\}] \leq \frac{n! e^n}{n^n \sqrt{1.6n\pi}} \exp\left\{-\frac{\frac{1}{2} n^2 \sum_{j \notin \mathcal{W}_\eta} \nu_j^2 (1 - e^{-2\theta})(1 + o^\eta(1))}{1 + \sum_{j \in \mathcal{W}_\eta} \nu_j (D_j - 1)}\right\} (1 + o^\eta(1)).$$

Substituting (66) and Stirling's formula into the right-hand side of the above inequality leads to

$$\mathbb{E}_\nu[\exp\{\theta(S_n^* - n)\}] \leq \frac{1}{\sqrt{0.8}} \exp\left\{-\frac{1}{2} n^2 \left(\sum_{j \notin \mathcal{W}_\eta} \nu_j^2\right) (1 - e^{-2\theta})(1 + o^\eta(1))\right\} (1 + o(1)).$$

Taking logarithm on both sides gives the claim of this proposition. ■



### C. Proof of Theorem 1 and Theorem 3

*Proof of Theorem 3:* Let  $\Lambda_{\mathbf{q}}(\theta)$  be the limit of the logarithmic moment generating function of  $\Lambda_{q^{(n)}, \tilde{S}_n^*}$ :

$$\Lambda_{\mathbf{q}}(\theta) := \lim_{n \rightarrow \infty} \frac{m}{n^2} \Lambda_{q^*, \tilde{S}_n^*}(\theta).$$

It follows from Proposition 4 that the limit exists and is given by the following  $C^1$  function:

$$\Lambda_{\mathbf{q}}(\theta) = \frac{1}{2}(e^{-2\theta} - 1)\kappa(\mathbf{q}).$$

Denote its Fenchel-Legendre transformation

$$\Lambda_{\mathbf{q}}^*(t) := \sup_{\theta} [\theta t - \Lambda_{\mathbf{q}}(\theta)].$$

It follows from the Gärtner-Ellis Theorem [46, Theorem 2.3.6] that

$$\begin{aligned} & -\limsup_{n \rightarrow \infty} \frac{m}{n^2} \log(\mathbb{P}_{q^{(n)}}\{S_n^* \leq \mathbb{E}_p[S_n^*] + \frac{n^2}{m}\tau\}) \\ &= -\limsup_{n \rightarrow \infty} \frac{m}{n^2} \log(\mathbb{P}_{q^{(n)}}\{\tilde{S}_n^* \geq -\mathbb{E}_p[S_n^*] - n - \frac{n^2}{m}\tau\}) \\ &= \inf_{t \geq -\tau-1} \Lambda_1^*(t) = \Lambda_1^*(-\tau-1) \\ &= \sup_{\theta \geq 0} \{\theta(-1-\tau) - \frac{1}{2}(e^{-2\theta} - 1)\kappa(\mathbf{q})\}. \end{aligned}$$

where  $-\tau-1$  is the normalized limit of  $-\mathbb{E}_p[S_n^*] - n - \frac{n^2}{m}\tau$  by Lemma 7. ■

*Proof of Theorem 1:* The proof for the result on the generalized error exponent of false alarm  $J_F(\phi^*)$  is very similar to that of Theorem 3. Let  $\Lambda_0(\theta)$  be the limit of the logarithmic moment generating function of  $\Lambda_{p, \tilde{S}_n^*}$ :

$$\Lambda_0(\theta) := \lim_{n \rightarrow \infty} \frac{m}{n^2} \Lambda_{p, \tilde{S}_n^*}(\theta).$$

It follows from Proposition 4 that the limit exists and is given by the following  $C^1$  function:

$$\Lambda_0(\theta) = \frac{1}{2}(e^{-2\theta} - 1).$$

Let  $\Lambda_0^*(t) = \sup_{\theta} [\theta t - \Lambda_0(\theta)]$ . It follows from the Gärtner-Ellis Theorem that

$$\begin{aligned} & -\limsup_{n \rightarrow \infty} \frac{m}{n^2} \log(\mathbb{P}_p(\phi_n^* = 1)) \\ &= -\limsup_{n \rightarrow \infty} \frac{m}{n^2} \log(\mathbb{P}_p\{\tilde{S}_n^* \leq -\mathbb{E}_p[S_n^*] - n - \frac{n^2}{m}\tau\}) \\ &= \inf_{t \leq -\tau-1} \Lambda_0^*(t) = \Lambda_0^*(-\tau-1) \\ &= \sup_{\theta} \{\theta(-\tau-1) - \frac{1}{2}(e^{-2\theta} - 1)\} = J_F^*(\tau). \end{aligned}$$

For the result on the generalized error exponent of missed detection  $J_M(\phi^*)$ , we prove an upper-bound and a lower-bound. For the upper-bound, consider the sequence of distributions given in (21) and (22) and let  $\mathbf{q}^*$  denote this sequence. The rate function associated with  $\mathbf{q}^*$  satisfies

$$J_{\mathbf{q}^*}(\phi^*, \tau) = J_M^*(\tau).$$

On the other hand, since each element of  $\mathbf{q}^*$  is in the set of alternative distributions, it follows from the definition of  $J_M(\phi^*)$  and  $J_{\mathbf{q}^*}(\phi^*, \tau)$  that

$$J_M(\phi^*) \leq J_{\mathbf{q}^*}(\phi^*, \tau)$$

To obtain the lower-bound on  $J_M(\phi^*)$ , we apply Proposition 5 and Proposition 6. We only need to prove it for the case  $\tau \in [0, \underline{\kappa}(\varepsilon))$ . The case  $\tau = \underline{\kappa}(\varepsilon)$  will then follow from a continuity argument.

Take  $\theta_0$  to be the maximizer in the optimization problem defining  $J_M^*(\tau)$  (see (16)). It is not difficult to see that  $\theta_0 > 0$ . It follows from Lemma 1 that

$$m \sum_{j \notin \mathcal{W}_\eta} q_j^2 \geq (1 + \underline{\kappa}(\frac{\varepsilon - \beta(q)}{1 - \beta(q)}))(1 - \beta(q))(1 + o(1)).$$

Thus, for any  $\delta > 0$ , we can choose  $\eta, \beta_0$  small enough so that for any  $q \in \mathcal{Q}_n$  satisfying  $\beta(q) \leq \beta_0$ , we have  $m \sum_{j \notin \mathcal{W}_\eta} q_j^2 \geq (1 + \underline{\kappa}(\varepsilon))(1 - \delta)$ . It then follows from Proposition 6 that for large enough  $n$ ,

$$\Lambda_{q, \tilde{S}_n^*}(\theta_0) \leq \frac{1}{2} \frac{n^2}{m} (1 + \underline{\kappa}(\varepsilon))(e^{-2\theta_0} - 1)(1 - \delta)^2 + O(1). \quad (70)$$

For  $q$  satisfying  $\beta(q) \geq \beta_0$ , it follows from Proposition 5 that for large enough  $n$ ,

$$\Lambda_{q, \tilde{S}_n^*}(\theta_0) \leq -\beta_0 \alpha(\theta_0) n. \quad (71)$$

We can pick  $n$  large enough so that the right-hand side of (71) is smaller than the right-hand side of (70). Applying the Chernoff bound leads to

$$\begin{aligned} & \log(\sup_{q \in \mathcal{Q}_n} P_q(\phi_n^* = 0)) \\ & \leq -\theta_0(\mathbb{E}_p[\tilde{S}_n^*] - \tau_n) + \sup_{q \in \mathcal{Q}_n} \Lambda_{q, \tilde{S}_n^*}(\theta_0) \\ & \leq \theta_0(\tau_n - \mathbb{E}_p[\tilde{S}_n^*]) + \frac{1}{2} \frac{n^2}{m} (1 + \underline{\kappa}(\varepsilon))(e^{-2\theta_0} - 1)(1 - \delta)^2 + O(1). \end{aligned}$$

Thus,

$$J_M(\phi^*) \geq \theta_0(-1 - \tau) - \frac{1}{2}(e^{-2\theta_0} - 1)(1 + \underline{\kappa}(\varepsilon))(1 - \delta)^2.$$

This holds for any  $\delta > 0$ . Consequently,  $J_M(\phi^*) \geq J_M^*(\tau)$ . ■

## APPENDIX C

### PROOFS OF THEOREM 4 AND THEOREM 5

#### A. Proof of Theorem 4

The performance of  $\phi^{*+}$  is analyzed by connecting it to the performance of  $\phi^*$ . We first show that its probability of missed detection is no larger than that of  $\phi^*$ . We then apply a result similar to Proposition 4 to analyze its probability of false alarm. Consider the statistic

$$\tilde{S}_n^{*+} = -S_n^{*+} - n.$$

Define

$$\Lambda_{\nu, \tilde{S}_n^{*+}}(\theta) := \log(\mathbb{E}_\nu[\exp\{\theta(\tilde{S}_n^{*+})\}]). \quad (72)$$

**Proposition 7.** *For any  $\nu \in \mathcal{P}_m^b$ , the logarithmic moment generating function for the statistic  $\tilde{S}_n^{*+}$  has the following asymptotic expansion*

$$\Lambda_{\nu, \tilde{S}_n^{*+}}(\theta) = \frac{n^2}{m} \left( m \sum_{j=1}^m \nu_j^2 \right) \left\{ -\theta + \frac{1}{2}[e^{-2\theta} - (1 - 2\theta)] \right\} + O\left(\frac{n^3}{m^2}\right) + O(1). \quad (73)$$

*Proof of Proposition 7:* The proof follows exactly the same step as that of Proposition 4 except some of the approximations are different. We now only describe the key steps and highlight the difference: First, the estimate of the saddle point is the same as (41) and (42). Second, different from (43), we have the following expression of the moment generating function:

$$\mathbb{E}_\nu[\exp\{\theta(\tilde{S}_n^{*+})\}] = \frac{n!}{2\pi} \lambda_0^{-n} e^{-\theta n} \operatorname{Re} \left[ \int_{-\pi}^{\pi} h(\psi) d\psi \right].$$

where instead of (44),

$$h(\psi) := e^{-in\psi} \prod_{j=1}^m (\lambda_0 \nu_j (e^\theta - 1) e^{i\psi} + e^{\lambda_0 \nu_j e^{i\psi}} + \sum_{l=2}^{\bar{l}} \frac{(\lambda_0 \nu_j)^l}{l!} (e^{\theta \nu_l} - 1)).$$

It follows from  $\lambda_0 = n^{-\theta}(1 + o(1))$  that the last term is negligible when  $v_2 = 0$  and  $\bar{l} < \infty$ .

$$\sum_{l=2}^{\bar{l}} \frac{(\lambda_0 \nu_j)^l}{l!} (e^{\theta \nu_l} - 1) = O\left(\frac{n^3}{m^3}\right)$$

The asymptotic approximation of  $h(\psi)$  is the same as that in (45):

$$h(\psi) = e^{-in\psi} \prod_{j=1}^m (\lambda_0 \nu_j (e^\theta - 1) e^{i\psi} + 1 + \lambda_0 \nu_j e^{i\psi} + O(\frac{n^2}{m^2})).$$

Finally, the approximations of  $H(0), H'(0), H''(\psi)$  are the same as in (48). Therefore,  $\Lambda_{\nu, \tilde{S}_n^{*+}}$  has the same asymptotic approximation as that of  $\Lambda_{\nu, \tilde{S}_n^*}$  up to an approximation error of  $O(\frac{n^3}{m^2})$ . ■

*Proof of Theorem 4:* Since  $v_l \geq 0$  for  $l \geq 2$ , we have

$$S_n^{*+} \geq S_n^*.$$

Thus, for the same sequence of thresholds  $\tilde{\tau}_n$ , we have

$$\mathbb{P}_q\{S_n^{*+} \leq \tilde{\tau}_n\} \leq \mathbb{P}_q\{S_n^* \leq \tilde{\tau}_n\}$$

On the other hand, since  $\Lambda_{\nu, \tilde{S}_n^{*+}}$  has the same asymptotic approximation as that of  $\Lambda_{\nu, \tilde{S}_n^*}$  up to an approximation error of  $O(\frac{n^3}{m^2})$ , we have

$$\begin{aligned} & \log \mathbb{P}_p\{S_n^{*+} \geq -n + \tilde{\tau}_n\} \\ &= \log \mathbb{P}_p\{\tilde{S}_n^{*+} \leq -\tilde{\tau}_n\} \\ &\leq \theta(-\tilde{\tau}_n) + \Lambda_{p, \tilde{S}_n^{*+}}(-\theta) \\ &= -\theta \tilde{\tau}_n + \frac{n^2}{m} \left( \theta + \frac{1}{2} [e^{2\theta} - (1 + 2\theta)] \right) + O\left(\frac{n^3}{m^2}\right) + O(1). \end{aligned}$$

which is the same bound as that for  $\log \mathbb{P}_p\{S_n^* \geq -n + \tilde{\tau}_n\}$ . ■

### B. Proof of Theorem 5

The proof of Theorem 5 follows exactly the same steps as those in the proof of Theorem 1. We use Proposition 8, Proposition 9 and Proposition 10 in place of Proposition 4, Proposition 5 and Proposition 6.

Denote

$$\Lambda_{\nu, S_n^w}(\theta) := \log(\mathbb{E}_\nu[\exp\{\theta S_n^w\}]).$$

**Proposition 8.** For any  $\nu \in \mathcal{P}_m^b$ , the logarithmic moment generating function for the statistic  $S_n^w$  has the following asymptotic expansion

$$\Lambda_{\nu, S_n^w}(\theta) = \frac{1}{2} n^2 \left( \sum_{j=1}^m (p_j - \nu_j)^2 \right) \theta + \frac{1}{2} n^2 \left( \sum_{j=1}^m \nu_j^2 \right) [e^\theta - (1 + \theta)] + O\left(\frac{n^3}{m^2}\right) + O(1).$$

**Proposition 9.** For all sufficiently small  $\eta > 0$ , any  $\theta \in [-1, 0)$  and any  $\underline{\beta} > 0$ . There exists  $n_0$  such that for any  $n > n_0$ , and any  $\nu$  satisfying  $\beta(\nu) \leq \underline{\beta}$ , the following holds,

$$\Lambda_{\nu, S_n^w}(\theta) \leq -\beta(q) \alpha'(\theta) n$$

where  $\alpha'(\theta) > 0$  for  $\theta \in [-1, 0)$ .

**Proposition 10.** For any  $\delta > 0$ ,  $\theta \in [-1, 0)$ ,  $\bar{\eta} > 0$ , there exists  $\eta \in (0, \bar{\eta})$ ,  $\bar{\beta} > 0$ , and  $n_0$  such that for any  $n > n_0$ , and any  $\nu$  satisfying  $\beta(q) \leq \bar{\beta}$ , the following holds,

$$\Lambda_{\nu, S_n^w}(\theta) \leq \frac{n^2}{m} \left[ \left( m \sum_{j \notin \mathcal{W}_\eta(\nu)} (p_j - \nu_j)^2 \right) \theta + \frac{1}{2} \left( m \sum_{j \notin \mathcal{W}_\eta(\nu)} \nu_j^2 \right) (e^\theta - (1 + \theta)) \right] (1 - \delta).$$

We only outline the proof for Proposition 8.

*Proof of Proposition 8:* The steps are the same as thos in the proof of Proposition 4. Again, we describe the main steps and highlight the difference. First, the estimate of the saddle point is different than that in (41) and (42). We have

$$\begin{aligned} \lambda_0 &= n(1 + w), \\ w &= n \left( \sum_j \nu_j p_j \theta - \sum_j \nu_j^2 (e^\theta - 1) \right) \left( 1 + O\left(\frac{n}{m}\right) \right). \end{aligned}$$

Second, different from (43), we have the following expression of the moment generating function:

$$\mathbb{E}_\nu[\exp\{\theta S_n^w\}] = \frac{n!}{2\pi} \lambda_0^{-n} \operatorname{Re} \left[ \int_{-\pi}^{\pi} h(\psi) d\psi \right]$$

where

$$\begin{aligned} h(\psi) &= e^{-in\psi} \prod_{j=1}^m [\exp\{\lambda_0 \nu_j e^{i\psi}\} + (e^{\frac{1}{2}n^2 p_j^2 \theta} - 1) + \lambda_0 e^{i\psi} \nu_j (e^{-np_j \theta} - 1) + \frac{1}{2} \lambda_0^2 e^{2i\psi} \nu_j^2 (e^\theta - 1)] \\ &= e^{-in\psi} \exp\{ne^{i\psi} + O\left(\frac{n^2}{m}\right)\}, \end{aligned}$$

Finally, the approximation of  $\operatorname{Re}(H(0))$  is different from that in (48)

$$\operatorname{Re}(H(0)) = n(1 + w) + \frac{1}{2}n^2 \left( \sum_{j=1}^m (p_j - \nu_j)^2 \right) \theta + \frac{1}{2}n^2 \left( \sum_{j=1}^m \nu_j^2 \right) (e^\theta - 1 - \theta) + O\left(\frac{n^3}{m^2}\right).$$

The rest of the steps are the same as those in Proposition 4. ■

*Proof of Theorem 5:* We first prove the lower-bound on  $J_F(\phi^w)$ . Substituting the asymptotic approximation of  $\Lambda_{p, S_n^w}(\theta)$  given in Proposition 8 into the Chernoff bound, we obtain that for  $\theta \geq 0$ ,

$$\begin{aligned} &\log P_p(\phi_n^w = 1) \\ &\leq -\theta \tau_n + \Lambda_{p, S_n^w}(\theta) \\ &= -\theta \tau_n + n^2 \left( \sum_{j=1}^m p_j^2 \right) \frac{1}{2} [e^\theta - (1 + \theta)] + O\left(\frac{n^3}{m^2}\right) + O(1). \end{aligned}$$

Since  $m \sum_{j=1}^m p_j^2 \leq \gamma^2$ , which is a consequence of Assumption 2, we have

$$J_F(\phi^w) \geq \sup_{\theta \geq 0} \left\{ \frac{1}{2} \tau \theta - \frac{1}{2} \gamma^2 [e^\theta - (1 + \theta)] \right\} > 0.$$

Lower-bounding  $J_M(\phi^w)$  requires us to obtain a uniform bound on the probability  $P_q(\phi_n = 0)$  over  $q \in \mathcal{Q}_n$ . We apply Proposition 9 and Proposition 10. Using an argument similar to the proof in Theorem 1, we conclude that for any  $\delta > 0$ , and  $\theta \in (0, 1]$ , for large enough  $n$ ,

$$\begin{aligned} &\log P_q(\phi_n^w = 0) \\ &\leq \theta \tau_n + \Lambda_{q, S_n^w}(-\theta) \\ &= \theta \tau_n - \frac{n^2}{m} \left[ \frac{1}{2} \theta m \sum_{j=1}^m (q_j - p_j)^2 - \frac{1}{2} \left( m \sum_{j=1}^m q_j^2 \right) (e^{-\theta} - (1 - \theta)) \right] (1 - \delta). \end{aligned}$$

We need to upper-bound the right-hand side uniformly over all  $q \in \mathcal{Q}_n$ . Using the inequalities  $q_j^2 \leq 2p_j^2 + 2(p_j - q_j)^2$  and  $e^{-\theta} - (1 - \theta) \leq \frac{1}{2}\theta^2$  for  $\theta > 0$ , we obtain

$$\begin{aligned} & \frac{m}{n^2} \log \mathbb{P}_q(\phi_n^W = 0) \\ & \leq \theta \frac{m\tau_n}{n^2} - [\frac{1}{2}\theta m \sum_{j=1}^m (q_j - p_j)^2 - \frac{1}{2}\theta^2 (m \sum_{j=1}^m p_j^2 + m \sum_{j=1}^m (q_j - p_j)^2)](1 - \delta) + O(1) \\ & = \frac{1}{2}\theta [-(m \sum_{j=1}^m (q_j - p_j)^2)(1 - \theta) + \theta(m \sum_{j=1}^m p_j^2)](1 - \delta) + \theta \frac{m\tau_n}{n^2} + O(1). \end{aligned}$$

Applying  $m \sum_{j=1}^m (q_j - p_j)^2 \geq 4\varepsilon^2$  and  $m \sum_{j=1}^m p_j^2 \leq \gamma^2$  leads to,

$$\frac{m}{n^2} \log[P_M(\phi_n^W)] \leq \frac{1}{2}\theta[-4\varepsilon^2(1 - \theta) + \theta\gamma^2](1 - \delta) + \frac{m\tau_n}{n^2} + O(1).$$

Taking  $\theta = (4\varepsilon^2(1 - \delta) - 2\tau)/[(8\varepsilon^2 + 2\gamma^2)(1 - \delta)]$ , and taking the limit on both sides gives

$$J_M(\phi^W) \geq \frac{1}{4}4\varepsilon^2 \frac{4\varepsilon^2(1 - \delta) - 2\tau}{(8\varepsilon^2 + 2\gamma^2)(1 - \delta)}.$$

Since this holds for all  $\delta > 0$ , and  $2\tau < 4\varepsilon^2$ , we conclude that

$$J_M(\phi^W) \geq \frac{1}{4}4\varepsilon^2 \frac{2\varepsilon^2 - \tau}{(8\varepsilon^2 + 2\gamma^2)(\frac{1}{2} + \tau/(4\varepsilon^2))} > 0.$$

■

#### APPENDIX D PROOF OF THEOREM 2

We first give an outline of the proof: Consider any  $\tau \in [0, \kappa(\varepsilon)]$ . Given  $\delta > 0$ , a sequence of events  $\{B_{n,\tau,\delta}\}$  is constructed so that the following is satisfied:

(i) The probability of the event is close to the probability of false alarm:

$$\limsup_{n \rightarrow \infty} -\frac{m}{n^2} \log(\mathbb{P}_p(B_{n,\tau,\delta})) \leq J_F^*(\tau) - \delta. \quad (74)$$

(ii) For any  $z_1^n$  satisfying  $\{Z_1^n = z_1^n\} \subseteq B_{n,\tau,\delta}$ , the following uniform bound on the likelihood ratio holds:

$$\sup_{q \in \mathcal{Q}_n} \frac{q^n}{p^n}(z_1^n) \geq \exp\{-\frac{n^2}{m}(J_M^*(\tau) - J_F^*(\tau) + \delta)\}. \quad (75)$$

The lower-bound on  $P_M$  is then obtained from the following inequality:

$$\begin{aligned} P_M(\phi_n) & \geq \sup_{q \in \mathcal{Q}_n} \mathbb{P}_q(\{\phi_n = 0\} \cap B_{n,\tau,\delta}) \\ & \geq \sup_{q \in \mathcal{Q}_n} \frac{q^n}{p^n}(\{\phi_n = 0\} \cap B_{n,\tau,\delta}) \mathbb{P}_p(\{\phi_n = 0\} \cap B_{n,\tau,\delta}) \\ & \geq \sup_{q \in \mathcal{Q}_n} \frac{q^n}{p^n}(\{\phi_n = 0\} \cap B_{n,\tau,\delta}) (\mathbb{P}_p(B_{n,\tau,\delta}) - \mathbb{P}_p(\{\phi_n = 1\})). \end{aligned} \quad (76)$$

The first term on the right-hand side is lower-bounded in (75). The second term can be shown to have the same large deviations limit as that of  $\mathbb{P}_p(B_{n,\tau,\delta})$ :

$$\mathbb{P}_p(\{\phi_n = 0\} \cap B_{n,\tau,\delta}) \geq \mathbb{P}_p(B_{n,\tau,\delta}) - \mathbb{P}_p(\{\phi_n = 1\}) \quad (77)$$

The inequality in (74) ensures that  $\mathbb{P}_p(\{\phi_n = 1\})$  is negligible comparing to  $\mathbb{P}_p(B_{n,\tau,\delta})$ .

The technique of using uniform lower-bounds on likelihood ratio (LR) to prove lower-bounds of probability of missed detection has been applied in [5, 3]: In this prior work, a uniform bound on LR is obtained *over all possible*  $z_1^n$ . To prove the tight hardness result as in Theorem 2, we require the bound on LR to hold uniformly for the

sequences in the event  $B_n$ , instead of all sequences. This gives us the freedom to optimize  $B_n$  to obtain the tightest bound.

The technique to prove (75) has been previously used in providing hardness results for composite and hypothesis testing problems [5, 3, 35]. First, construct a collection of distributions so that for each distribution  $q$ , the likelihood ratio  $q/p$  has a simple expression. Second, show that for all observations  $\mathbf{z}_1^n := \{z_1, \dots, z_n\}$  in the event  $B_n$ , the average of  $\mathbb{P}_q\{\mathbf{Z}_1^n = \mathbf{z}_1^n\}/\mathbb{P}_p\{\mathbf{Z}_1^n = \mathbf{z}_1^n\}$  over the collection of distributions can be lower-bounded, which in turn lower-bounds the left-hand side of (75). The proof for  $\varepsilon < 0.5$  and  $\varepsilon \geq 0.5$  uses different constructions of distributions.

We now carry out these two steps: Construction of the event  $B_{n,\tau,\delta}$  and lower-bounding the likelihood ratio.

#### A. Construction of $B_{n,\tau,\delta}$

Define the event

$$B_{n,\tau,\delta} = \left\{ \sum_{j=1}^m \mathbb{I}\{n\Gamma_j^n = 1\} \geq n - (1 + \tau + \delta) \frac{n^2}{m}, \quad \sum_{j=1}^m \mathbb{I}\{n\Gamma_j^n = 2\} \geq \frac{1}{2}(1 + \tau - \delta) \frac{n^2}{m} \right\}. \quad (78)$$

The probability of the event  $B_{n,\tau,\delta}$  has the following asymptotic approximation:

**Lemma 10.** For  $\tau = 0$  and any  $\delta > 0$ ,

$$\lim_{n \rightarrow \infty} \mathbb{P}_p(B_{n,\tau,\delta}) = 1. \quad (79)$$

For any  $\tau, \delta$  satisfying  $\tau > \delta > 0$ ,

$$\lim_{n \rightarrow \infty} -\frac{m}{n^2} \log \mathbb{P}_p(B_{n,\tau,\delta}) = J_F^*(\tau - \delta). \quad (80)$$

*Proof of Lemma 10:* First consider the case where  $\tau = 0$ . Applying Theorem 1 with  $\tau$  replaced by  $\delta$  gives

$$\mathbb{P}_p\left\{ \sum_{j=1}^m \mathbb{I}\{n\Gamma_j^n = 1\} \leq n - (1 + \delta) \frac{n^2}{m} \right\} = 1 - o(1). \quad (81)$$

The following asymptotic approximations on the expectation and variance of the statistic  $\sum_{j=1}^m \mathbb{I}\{n\Gamma_j^n = 2\}$  follows from Lemma 6 and Lemma 8:

$$\begin{aligned} \mathbb{E}_p\left[ \sum_{j=1}^m \mathbb{I}\{n\Gamma_j^n = 2\} \right] &= \frac{1}{2} \frac{n^2}{m} (1 + o(1)), \\ \text{Var}_p\left[ \sum_{j=1}^m \mathbb{I}\{n\Gamma_j^n = 2\} \right] &= \frac{1}{2} \frac{n^2}{m} (1 + o(1)). \end{aligned}$$

Applying Chebyshev's inequality leads to

$$\mathbb{P}_p\left\{ \sum_{j=1}^m \mathbb{I}\{n\Gamma_j^n = 2\} \leq \frac{1}{2} \frac{n^2}{m} (1 - \delta) \right\} = O\left(\frac{m}{n^2}\right).$$

The claim of this lemma for  $\tau = 0$  follows from combining this inequality with (81).

Next consider the case where  $\tau > 0$ . We first obtain a large deviations characterization of

$$S^{(2)} := \sum_{j=1}^m \mathbb{I}\{n\Gamma_j^n = 2\}$$

by deriving an approximation to the logarithmic moment generating function. The steps are the same as those in the proof of Proposition 4. Again, we describe the main steps and highlight the difference. First, the estimate of the saddle point is different than that in (41) and (42). We have

$$\begin{aligned} \lambda_0 &= n(1 + w), \\ w &= -n \sum_j \nu_j^2 (e^\theta - 1) (1 + O(\frac{n}{m})). \end{aligned}$$

Second, different from (43), we have the following expression of the moment generating function:

$$\mathbb{E}_\nu^n[\exp\{\theta S^{(2)}\}] = \frac{n!}{2\pi} \lambda_0^{-n} \operatorname{Re}[\int_{-\pi}^{\pi} h(\psi) d\psi]$$

where

$$h(\psi) = e^{-in\psi} \prod_{j=1}^m [\exp\{\lambda_0 \nu_j e^{i\psi}\} + \frac{1}{2} \lambda_0^2 e^{2i\psi} \nu_j^2 (e^\theta - 1)] = e^{-in\psi} \exp\{n e^{i\psi} + O(\frac{n^2}{m})\}$$

Finally, the approximation of  $\operatorname{Re}(H(0))$  is different from that in (48)

$$\operatorname{Re}(H(0)) = n(1 + w) + \frac{1}{2} n^2 (\sum_{j=1}^m \nu_j^2) (e^\theta - 1) + O(\frac{n^3}{m^2}).$$

The rest of the steps are the same as those in Proposition 4. We obtain

$$\Lambda_{\nu, S^{(2)}}(\theta) = \frac{1}{2} \frac{n^2}{m} (m \sum_{j=1}^m \nu_j^2) (e^{-2\theta} - 1) + O(\frac{n^3}{m^2}) + O(1). \quad (82)$$

Applying the same steps as those for the characterization of  $J_F(\phi^*)$  in Theorem 1, we have

$$\lim_{n \rightarrow \infty} -\frac{m}{n^2} \log \mathbb{P}_p \left\{ \sum_{j=1}^m \mathbb{I}\{n \Gamma_j^n = 2\} \geq \frac{1}{2} (1 + \tau - \delta) \frac{n^2}{m} \right\} = J_F^*(\tau - \delta).$$

Applying Theorem 1 with  $\tau$  replaced by  $\tau + \delta$ , we obtain

$$\lim_{n \rightarrow \infty} -\frac{m}{n^2} \log \mathbb{P}_p \left\{ \sum_{j=1}^m \mathbb{I}\{n \Gamma_j^n = 1\} \leq n - (1 + \tau + \delta) \frac{n^2}{m} \right\} = J_F^*(\tau + \delta).$$

Note that  $J_F^*(\tau + \delta) > J_F^*(\tau - \delta)$ . Thus the probability that the first constraint in the definition of  $B_{n, \tau, \delta}$  is violated is negligible comparing to the probability that the second constraint is satisfied. This shows that the probability of  $B_{n, \tau, \delta}$  can be approximated by the probability that the second constraint in the definition of  $B_{n, \tau, \delta}$  is satisfied. This leads to the claim of the lemma. ■

#### B. A lower-bound on the likelihood ratio for $\varepsilon \geq 0.5$

When  $\varepsilon \geq 0.5$ , we use the following construction of distributions: Let  $U_m$  denote the collection of all subsets of  $[m]$  whose cardinality is  $\lfloor m(1 - \varepsilon) \rfloor$ . For each  $\mathcal{U} \in U_m$ , define the distribution

$$q_{\mathcal{U}, j} = \begin{cases} \frac{1}{\lfloor m(1 - \varepsilon) \rfloor}, & j \in \mathcal{U}; \\ 0, & j \in [m] \setminus \mathcal{U}. \end{cases}$$

Consider the mixture  $\bar{q}^n = \frac{1}{|U_m|} \sum_{\mathcal{U} \in U_m} q_{\mathcal{U}}^n$ . The following bound on  $\bar{q}^n/p^n$  holds:

**Lemma 11.** Suppose  $\varepsilon \geq 0.5$ . For any sequence  $\mathbf{z}_1^n = \{z_1, \dots, z_n\}$  satisfying  $\{\mathbf{Z}_1^n = \mathbf{z}_1^n\} \subseteq B_{n, \tau, \delta}$ , the following holds:

$$\log\left(\frac{\bar{q}^n}{p^n}(\mathbf{z}_1^n)\right) \geq -\frac{1}{2} \frac{n^2}{m} [\underline{\kappa}(\varepsilon) - \log(1 + \underline{\kappa}(\varepsilon))(1 + \tau - \delta)] + O(\frac{n^3}{m^2}).$$

*Proof of Lemma 11:* Let  $\mathcal{S} := \{j : j \text{ appears in } \mathbf{z}_1^n\}$ . Let  $s = |\mathcal{S}|$ . It follows from  $\{\mathbf{Z}_1^n = \mathbf{z}_1^n\} \subseteq B_{n, \tau, \delta}$  that

$$n - \frac{1}{2} \frac{n^2}{m} (1 + \tau + 3\delta) \leq s \leq n - \frac{1}{2} \frac{n^2}{m} (1 + \tau - \delta). \quad (83)$$

The likelihood ratio  $\frac{q_{\mathcal{U}}^n}{p^n}$  has the expression:  $\frac{q_{\mathcal{U}}^n}{p^n}(\mathbf{z}_1^n) = (\frac{m}{\lfloor m(1 - \varepsilon) \rfloor})^n \mathbb{I}_{\mathcal{S} \subseteq \mathcal{U}}$ . Thus,

$$\frac{\bar{q}^n}{p^n}(\mathbf{z}_1^n) = \left(\frac{m}{\lfloor m(1 - \varepsilon) \rfloor}\right)^n \left(\frac{1}{|U_m|} \sum_{\mathcal{U} \in U_m} \mathbb{I}_{\mathcal{S} \subseteq \mathcal{U}}\right), \quad (84)$$

where

$$\frac{1}{|U_m|} \sum_{\mathcal{U} \in U_m} \mathbb{I}_{S \subseteq \mathcal{U}} = \frac{\binom{m-s}{\lfloor m(1-\varepsilon) \rfloor - s}}{\binom{m}{\lfloor m(1-\varepsilon) \rfloor}}.$$

Stirling's formula gives

$$\frac{\binom{m-s}{\lfloor m(1-\varepsilon) \rfloor - s}}{\binom{m}{\lfloor m(1-\varepsilon) \rfloor}} = \left( \frac{\lfloor m(1-\varepsilon) \rfloor}{m} \right)^s \exp \left\{ -\frac{1}{2} \frac{s^2}{m} \frac{\varepsilon}{1-\varepsilon} + O\left(\frac{k^3}{m^2}\right) \right\} (1 + O(\frac{1}{m})).$$

Substituting this into (84) leads to

$$\frac{\bar{q}^n}{p^n}(\mathbf{z}_1^n) = (1-\varepsilon)^s \exp \left\{ -\frac{1}{2} \frac{s^2}{m} \frac{\varepsilon}{1-\varepsilon} + O\left(\frac{n^3}{m^2}\right) \right\} (1 + O(\frac{n}{m})).$$

The claim of this lemma follows from applying the inequality (83) and the fact that  $\underline{\kappa}(\varepsilon) = \frac{\varepsilon}{1-\varepsilon}$  when  $\varepsilon \geq 0.5$ . ■

### C. A lower-bound on the likelihood ratio for $\varepsilon < 0.5$

When  $\varepsilon < 0.5$ , we use the following construction of distributions: Let  $U_m$  denote the collection of all subsets of  $[m]$  whose cardinality is  $\lfloor m/2 \rfloor$ . For each set  $\mathcal{U} \in U_m$ , define the distribution  $q_{\mathcal{U}}$  as

$$q_{\mathcal{U},j} = \begin{cases} \frac{1}{m} + \frac{\varepsilon}{\lfloor m/2 \rfloor}, & j \in \mathcal{U}; \\ \frac{1}{m} - \frac{\varepsilon}{\lfloor m/2 \rfloor}, & j \in [m] \setminus \mathcal{U}. \end{cases}$$

This collection of distributions can be obtained by taking the worst-case distribution  $q^*$  given in (21), and permuting the symbols in the alphabet  $[m]$ .

Let  $q_{\mathcal{U}}^n$  be the  $n$ -order product of  $q_{\mathcal{U}}$ . Define the following mixture distribution,

$$\bar{q}^n = \frac{1}{|U_m|} \sum_{\mathcal{U} \in U_m} q_{\mathcal{U}}^n.$$

The LR  $\bar{q}^n/p^n$  can be lower-bounded on  $B_{n,\tau,\delta}$ :

**Lemma 12.** Suppose  $\varepsilon < 0.5$ . The following holds for any sequence  $\mathbf{z}_1^n$  satisfying  $\{\mathbf{Z}_1^n = \mathbf{z}_1^n\} \subseteq B_{n,\tau,\delta}$ :

$$\log\left(\frac{\bar{q}^n}{p^n}(\mathbf{z}_1^n)\right) \geq -\frac{n^2}{2m} [\underline{\kappa}(\varepsilon) - \log(1 + \underline{\kappa}(\varepsilon))(1 + \tau - \delta)](1 + o(1)) - \frac{n^2}{m} 2\delta \log(1 - 2\varepsilon).$$

*Proof of Lemma 12:* For simplicity of exposition we restrict to the case where  $m$  is even. Define

$$\begin{aligned} \mathcal{S}_1 &:= \{j : j \text{ appears in } \mathbf{z}_1^n \text{ exactly once}\}, \\ \mathcal{S}_2 &:= \{j : j \text{ appears in } \mathbf{z}_1^n \text{ exactly twice}\}. \end{aligned}$$

Let  $s_1 = |\mathcal{S}_1|$ ,  $s_2 = |\mathcal{S}_2|$ . It follows from  $\{\mathbf{Z}_1^n = \mathbf{z}_1^n\} \subseteq B_{n,\tau,\delta}$  that

$$n \geq s_1 \geq n - \frac{n^2}{m}(1 + \tau + \delta), \quad s_2 \geq \frac{1}{2} \frac{n^2}{m}(1 + \tau - \delta). \quad (85)$$

Consider any set  $\mathcal{U} \in U_m$ . Let  $k_{\mathcal{U},1} = |\mathcal{U} \cap \mathcal{S}_1|$ , and  $k_{\mathcal{U},2} = |\mathcal{U} \cap \mathcal{S}_2|$ . Then

$$\frac{q_{\mathcal{U}}^n}{p^n}(\mathbf{z}_1^n) \geq (1 - 2\varepsilon)^n \left( \frac{1 + 2\varepsilon}{1 - 2\varepsilon} \right)^{k_{\mathcal{U},1} + 2k_{\mathcal{U},2}}.$$

Consequently,

$$\frac{\bar{q}^n}{p^n}(\mathbf{z}_1^n) \geq G(s_1, s_2) \quad (86)$$

where

$$\begin{aligned} G(s_1, s_2) &:= \frac{1}{|U_m|} (1 - 2\varepsilon)^n \sum_{k_1=1}^{s_1} \sum_{k_2=1}^{s_2} \left( \left( \frac{1 + 2\varepsilon}{1 - 2\varepsilon} \right)^k \left( \frac{1 + 2\varepsilon}{1 - 2\varepsilon} \right)^{2k_2} |\{\mathcal{U} \in U_m : k_{\mathcal{U},1} = k_1, k_{\mathcal{U},2} = k_2\}| \right) \\ &= \frac{1}{\binom{m}{m/2}} (1 - 2\varepsilon)^n \sum_{k_1=1}^{s_1} \sum_{k_2=1}^{s_2} \left( \left( \frac{1 + 2\varepsilon}{1 - 2\varepsilon} \right)^k \left( \frac{1 + 2\varepsilon}{1 - 2\varepsilon} \right)^{2k_2} \binom{s_1}{k_1} \binom{s_2}{k_2} \binom{m - (s_1 + s_2)}{m/2 - (k_1 + k_2)} \right). \end{aligned} \quad (87)$$



The summand on the right-hand side of (87) takes its maximum value approximately when

$$k_1 = \bar{k}_1 := \lceil \frac{1+2\varepsilon}{2} s_1 \rceil, \quad k_2 = \bar{k}_2 := \lceil \frac{1}{2} (1 + \frac{4\varepsilon}{1+4\varepsilon^2}) \rceil. \quad (88)$$

We apply the Laplace method to approximate the summation: Denote

$$y(\Delta_1, \Delta_2) = \left( \frac{1+2\varepsilon}{1-2\varepsilon} \right)^{\bar{k}_1 + \Delta_1 + 2(\bar{k}_2 + \Delta_2)} \binom{s_1}{\bar{k}_1 + \Delta_1} \binom{s_2}{\bar{k}_2 + \Delta_2} \binom{m - (s_1 + s_2)}{m/2 - (\bar{k}_1 + \Delta_1 + \bar{k}_2 + \Delta_2)} / \binom{m}{m/2}.$$

Stirling's formula gives

$$\left( \frac{m - (s_1 + s_2)}{m/2 - (\bar{k}_1 + \Delta_1 + \bar{k}_2 + \Delta_2)} \right) / \binom{m - (s_1 + s_2)}{m/2 - (\bar{k}_1 + \Delta_1 + \bar{k}_2 + \Delta_2)} = \exp\{1 + O(\frac{(\Delta_1 + \Delta_2)(\bar{k}_1 + \bar{k}_2)}{m}) + o(1)\}. \quad (89)$$

Let

$$y_1(\Delta_1) = \left( \frac{1+2\varepsilon}{1-2\varepsilon} \right)^{\Delta_1} \binom{s_1}{\bar{k}_1 + \Delta_1} / \binom{s_1}{\bar{k}_1},$$

$$y_2(\Delta_2) = \left( \frac{1+2\varepsilon}{1-2\varepsilon} \right)^{2\Delta_2} \binom{s_2}{\bar{k}_2 + \Delta_2} / \binom{s_2}{\bar{k}_2}$$

Note that  $y(\bar{k}_1, \bar{k}_2)$  is the largest summand. Keeping only the  $\lceil \sqrt{s_1} \rceil \lceil \sqrt{s_2} \rceil$  number of terms in the summation in (86) whose index  $(k_1, k_2)$  is close to  $(\bar{k}_1, \bar{k}_2)$ , and applying (89), we obtain

$$\begin{aligned} \frac{\bar{q}^n}{p^n} (z_1^n) &\geq \sum_{\Delta_1 = -\lceil \sqrt{s_1} \rceil}^{\lceil \sqrt{s_1} \rceil} \sum_{\Delta_2 = -\lceil \sqrt{s_2} \rceil}^{\lceil \sqrt{s_2} \rceil} y(\Delta_1, \Delta_2) \\ &= \left( \sum_{\Delta_1 = -\lceil \sqrt{s_1} \rceil}^{\lceil \sqrt{s_1} \rceil} y_1(\Delta_1) \right) \left( \sum_{\Delta_2 = -\lceil \sqrt{s_2} \rceil}^{\lceil \sqrt{s_2} \rceil} y_2(\Delta_2) \right) y(0, 0) \exp\{1 + O(\frac{n^{\frac{3}{2}}}{m})\}. \end{aligned} \quad (90)$$

We first approximate  $\sum_{\Delta_1 = -\lceil \sqrt{s_1} \rceil}^{\lceil \sqrt{s_1} \rceil} y_1(\Delta_1)$ . Note that for  $\Delta_1 > 0$ ,

$$\log(y_1(\Delta_1)) = \Delta_1 \log\left(\frac{1+2\varepsilon}{1-2\varepsilon}\right) + \sum_{t=1}^{\Delta_1} \log\left(\frac{s - \bar{k}_1 - t}{\bar{k}_1 + t}\right).$$

Approximating the above summation by integrals leads to

$$\log(y_1(\Delta_1)) = -\frac{1}{2} \left( \frac{1}{s_1 - \bar{k}_1} + \frac{1}{\bar{k}_1} \right) \Delta_1^2 (1 + o(1)) + O(1).$$

Approximating the summation over  $\Delta_1$  using integrals, and applying the above approximation of  $y_1(\Delta_1)$  leads to

$$\sum_{\Delta_1 = -\lceil \sqrt{s_1} \rceil}^{\lceil \sqrt{s_1} \rceil} y_1(\Delta_1) = e^{O(1)} \int_{-\infty}^{\infty} e^{-\frac{1}{2} \left( \frac{1}{s_1 - \bar{k}_1} + \frac{1}{\bar{k}_1} \right) \Delta_1^2} d\Delta_1 = e^{O(1)} \sqrt{\frac{(s_1 - \bar{k}_1) \bar{k}_1}{s_1}} = e^{O(1)} \sqrt{s_1},$$

where the last equality follows from (88). A similar approximation for the summation over  $y_2$  holds:

$$\sum_{\Delta_2 = -\lceil \sqrt{s_2} \rceil}^{\lceil \sqrt{s_2} \rceil} y_2(\Delta_2) = e^{O(1)} \sqrt{s_2}.$$

Substituting these into (87) gives

$$G(s_1, s_2) = e^{O(1) + O(\frac{n^{3/2}}{m})} \sqrt{s_1 s_2} (1 - 2\varepsilon)^n \left( \frac{1+2\varepsilon}{1-2\varepsilon} \right)^{\bar{k}_1} \left( \frac{1+2\varepsilon}{1-2\varepsilon} \right)^{2\bar{k}_2} \binom{s_1}{\bar{k}_1} \binom{s_2}{\bar{k}_2} \binom{m - (s_1 + s_2)}{m/2 - (\bar{k}_1 + \bar{k}_2)} / \binom{m}{m/2}. \quad (91)$$

Stirling's formula gives the following asymptotic approximations the combinatorial terms in (91):

$$\begin{aligned}\binom{s_1}{\bar{k}_1} &= \frac{(1+2\varepsilon)^{-\bar{k}_1} (1-2\varepsilon)^{\bar{k}_1-s_1} 2^{s_1}}{\sqrt{2\pi \bar{k}_1 (s_1 - \bar{k}_1)/s_1}} (1+o(1)), \\ \binom{s_2}{\bar{k}_2} &= \frac{(1+2\varepsilon)^{-2\bar{k}_2} (1-2\varepsilon)^{2(\bar{k}_2-s_2)} (1+4\varepsilon^2)^{s_2} 2^{s_2}}{\sqrt{2\pi \bar{k}_2 (s_2 - \bar{k}_2)/s_2}} (1+o(1)), \\ \binom{m-(s_1+s_2)}{m/2-(\bar{k}_1+\bar{k}_2)} &= 2^{m-s_1-s_2} \exp\left\{-\frac{s_1^2(2\varepsilon)^2}{2m} (1+o(1))\right\} \frac{\sqrt{2}}{\sqrt{\pi m}} (1+o(1)), \\ \binom{m}{m/2} &= \frac{2^m}{\sqrt{2\pi m}} (1+o(1)).\end{aligned}$$

Substituting these approximations and the value of  $\bar{k}_1$  and  $\bar{k}_2$  into (91) leads to

$$G(s_1, s_2) = (1-2\varepsilon)^{n-s_1-2s_2} \exp\left\{-\frac{s_1^2(2\varepsilon)^2}{2m} (1+o(1)) + s_2 \log(1+4\varepsilon^2)\right\} \exp\{O(1) + O(\frac{n^{3/2}}{m})\}.$$

Combining this with (85), (86) gives the claim of the lemma. ■

#### D. Proof of Theorem 2

*Proof:* Consider first the case  $\tau > 0$ . Consider any  $\delta \in (0, \tau)$ , and any test  $\phi$  such that  $J_F(\phi) \geq J_F^*(\tau)$ . Applying (77) and Lemma 10, we obtain

$$\lim_{n \rightarrow \infty} -\frac{m}{n^2} \log \mathbb{P}_p(\{\phi_n = 0\} \cap B_{n,\tau,\delta}) = J_F^*(\tau - \delta). \quad (92)$$

When  $\varepsilon \geq 0.5$ , we apply (76), (92), and Lemma 11 to obtain

$$\begin{aligned}J_M(\phi) &\leq \frac{1}{2}[\underline{\kappa}(\varepsilon) - \log(1 + \underline{\kappa}(\varepsilon))(1 + \tau - \delta)] + J_F^*(\tau - \delta) \\ &= J_M^*(\tau - \delta) + r_2(\delta).\end{aligned} \quad (93)$$

where  $r_2$  again vanishes as  $\delta \rightarrow 0$ ,

$$r_2(\delta) = \frac{1}{2}[-\delta \log(1 + \underline{\kappa}(\varepsilon)) + (1 + \tau) \log(1 - \frac{\delta}{1 + \tau}) - \delta \log(1 + \tau - \delta) + \delta].$$

We have used the following explicit expressions of  $J_F^*$  and  $J_M^*$ :

$$\begin{aligned}J_F^*(\tau) &= \frac{1}{2}[-\tau + (1 + \tau) \log(1 + \tau)], \\ J_M^*(\tau) &= \frac{1}{2}[\underline{\kappa}(\varepsilon) - \tau + (1 + \tau) \log(\frac{1 + \tau}{1 + \underline{\kappa}(\varepsilon)})].\end{aligned}$$

Since (93) holds for any  $\delta > 0$  and  $J_M^*(\tau)$  is continuous, we conclude  $J_M(\phi) \leq J_M^*(\tau)$ .

When  $\varepsilon < 0.5$ , we apply (76), (92), and Lemma 12 to obtain

$$\begin{aligned}J_M(\phi) &\leq \frac{1}{2}[\underline{\kappa}(\varepsilon) - \log(1 + \underline{\kappa}(\varepsilon))(1 + \tau - \delta) + 4\delta \log(1 - 2\varepsilon) + J_F^*(\tau - \delta)] \\ &= J_M^*(\tau - \delta) + r_1(\delta).\end{aligned} \quad (94)$$

where

$$r_1(\delta) = \frac{1}{2}[-\delta \log(1 + \underline{\kappa}(\varepsilon)) + (1 + \tau) \log(1 - \frac{\delta}{1 + \tau}) - \delta \log(1 + \tau - \delta) + \delta + 4\delta \log(1 - 2\varepsilon)].$$

Since the inequality (94) holds for any  $\delta > 0$ ,  $J_M^*(\tau)$  is continuous in  $\tau$ , and  $r_1(\delta) \rightarrow 0$  as  $\delta \rightarrow 0$ , we conclude that  $J_M(\phi) \leq J_M^*(\tau)$ .

The proof for the case where  $\tau = 0$  is exactly the same as that for the case  $\tau > 0$ , except (79) is used in place of (80). We omit the details. ■

## APPENDIX E

## PROOF OF LEMMA 3, LEMMA 4 AND LEMMA 5 USED IN THE PROOF OF THEOREM 6

*Proof of Lemma 3:* Applying Lemma 6 to the distribution  $q^* \in \mathcal{Q}_n$  given in (22) and (21) gives  $E_{q^*}[S_n^P] = E_p[S_n^P] + \frac{n^2}{m}\kappa(\varepsilon)(1 + o(1))$ . It follows from Chebyshev's inequality that for  $\tau_n > E_p[S_n^P] + \frac{n^2}{m}\underline{\kappa}(\varepsilon)$ ,

$$P_q^*\{\phi_n^P(\mathbf{Z}_1^n) = 1\} \leq \frac{\text{Var}_{q^*}[S_n^P]}{(\tau_n - E_p[S_n^P] - \frac{n^2}{m}\underline{\kappa}(\varepsilon))^2}.$$

Thus, in order for  $\lim_{n \rightarrow \infty} P_q^*\{\phi_n^P(\mathbf{Z}_1^n) = 1\} = 1$  to hold, we must have

$$(\tau_n - E_p[S_n^P] - \frac{n^2}{m}\underline{\kappa}(\varepsilon))^2 \leq \text{Var}_{q^*}[S_n^P](1 + o(1)) = 2\frac{n^2}{m}(1 + \underline{\kappa}(\varepsilon))(1 + o(1)).$$

where the last equality follows from Lemma 8. This leads to the claim of Lemma 3. ■

*Proof of Lemma 4:* Consider the statistic

$$\bar{S}_n^P = S_n^P - \frac{n}{m} \frac{(n\Gamma_1^n - np_1)^2}{np_1} = S_n^P - 2\frac{n^2}{m}\underline{\kappa}(\varepsilon) + O(\frac{n}{\sqrt{m}}).$$

The conditional distribution of  $\bar{S}_n^P$  in the event  $A$  under  $p$  is the same as the distribution of  $S_{n'}^P$  under  $p'$ , where the number of samples is  $n' = n - \lfloor \frac{n\sqrt{2\kappa(\varepsilon)}}{\sqrt{m}} \rfloor$  and  $p'$  is the uniform distribution over  $[m-1]$ . It then follows from Lemma 6 and Lemma 8 that

$$E_p[\bar{S}_n^P | A] = E_{p'}[S_{n'}^P] = n - \lfloor \frac{n\sqrt{2\kappa(\varepsilon)}}{\sqrt{m}} \rfloor + O(\frac{n^2}{m}),$$

$$\text{Var}_p[\bar{S}_n^P | A] = \text{Var}_{p'}[S_{n'}^P] = 2\frac{n^2}{m}(1 + o(1)).$$

It follows from Chebyshev's inequality, Lemma 6 and Lemma 8 that for large enough  $n$ ,

$$\begin{aligned} & P_p\{S_n^P \leq E_p[S_n^P] + \frac{n^2}{m}\underline{\kappa}(\varepsilon) + 2\frac{n}{\sqrt{m}} | A_n\} \\ &= P_p\{\bar{S}_n^P + 2\frac{n^2}{m}\underline{\kappa}(\varepsilon) \leq n + \frac{n^2}{m}\underline{\kappa}(\varepsilon) + 2\frac{n}{\sqrt{m}} + O(\frac{n}{\sqrt{m}}) | A_n\} \\ &= P_p\{\bar{S}_n^P \leq E_p[\bar{S}_n^P | A] - \frac{n^2}{m}\underline{\kappa}(\varepsilon) + O(\frac{n}{\sqrt{m}}) | A_n\} \\ &\leq \frac{2\frac{n^2}{m}(1 + O(\frac{n}{\sqrt{m}}))}{(\frac{n^2}{m}\underline{\kappa}(\varepsilon) + O(\frac{n}{\sqrt{m}}))^2} = O(\frac{m}{n^2}). \end{aligned}$$

*Proof of Lemma 5:* A simple combinatorial argument gives ■

$$P_p\{A_n\} = \binom{n}{\lfloor \frac{n\sqrt{2\kappa(\varepsilon)}}{\sqrt{m}} \rfloor} p_1^{\lfloor \frac{n\sqrt{2\kappa(\varepsilon)}}{\sqrt{m}} \rfloor} (1 - p_1)^{n - \lfloor \frac{n\sqrt{2\kappa(\varepsilon)}}{\sqrt{m}} \rfloor}.$$

Applying Stirling's formula and substituting  $p_1 = \frac{1}{m}$  leads to

$$P_p\{A_n\} = \exp\{-\frac{1}{2}\frac{n\sqrt{2\kappa(\varepsilon)}}{\sqrt{m}} \log(m)(1 + o(1))\}(1 + o(1)). \quad (95)$$

Since  $m = o(\frac{n^2}{\log(n)^2})$  and  $m = o(n^2)$ , we have

$$\frac{n\sqrt{2\kappa(\varepsilon)}}{\sqrt{m}} \log(m) = \frac{n\sqrt{2\kappa(\varepsilon)}}{\sqrt{m}} o(2 \log(n)) = o(\frac{n^2}{m}).$$

Substituting this into (95) leads to the claim of this lemma. ■

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